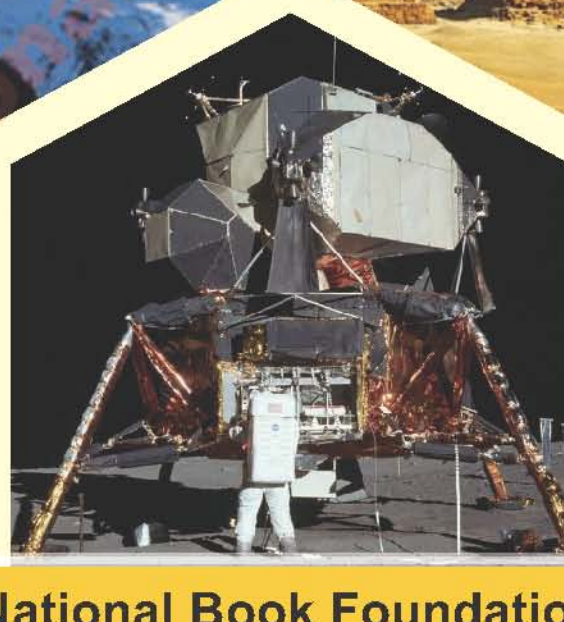
Textbook of

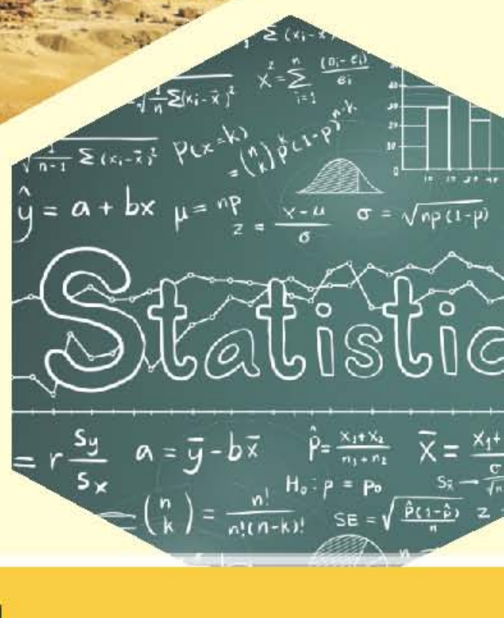
MATHEMATICS

Science Group

10

Based on National
Curriculum 2006





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Textbook of

Mathematics

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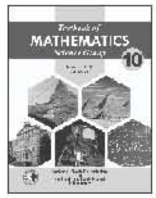
A Textbook of **Mathematics**
(*Science Group*) for Grade 10



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Preface

The Mathematics Science Group Grade 10 textbook is developed to cover the learning outcomes of the revised Mathematics National Curriculum 2006 comprehensively.

The Mathematics Science Group Grade 10 textbook builds a strong foundation of the subject through the use of well-researched and sound pedagogical principles. Adopting the popular Concrete- Pictorial-Abstract approach widely used in revised Mathematics Curriculum 2000, learners are introduced to new concepts through concrete manipulatives and engaging pictorials before they are led to see their abstract symbolic representations. This allows learners to have a deeper understating of key mathematical concepts, thus motivating them to learn.

Content is also clearly structured and spiralled across the levels to ensure a gradual build-up and review of skills as learner's progress up the grades. At the same time, emphasis is given on developing learners' problem-solving skills, critical thinking, as well as other 21st century skills.

The **Mathematics Science Group Grade 10** textbook also develops pupils' confidence for examinations through Check points, Exercises and Review Exercises for every concept. Every Chapter begins with a chapter opener which motivates learner's interest. Interesting real life situations and applications are used to develop learner's interest and curiosity so that they can appreciate the beauty of Mathematics in their environments.

The National Book Foundation endeavours to keep quality enhancement at the heart of textbooks development. Likewise, it strives to keep improving its textbooks by incorporation of feedback and suggestions from the students, teachers and the community Preface in subsequent editions of its new textbooks. Some errors pointed out as feedback which are rectified by authors in this edition. As always the National Book Foundation looks forward to receiving feedback on this new textbook for Mathematics Science Group Grade X to align it more closely to the needs of the learners.

Dr. Raja Mazhar Hameed

Managing Director

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

Contents

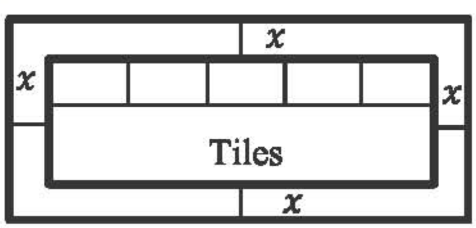
UNIT 01 Quadratic Equations

In this unit the students will be able to:

- Define quadratic equations.
- Solve quadratic equations by factorization and completing squares.
- Derive quadratic formula and use it to solve quadratic equations.
- Solve different types of equations which are reducible to quadratic form.
- Solve some types of radical equations.

You are building a rectangular tiles surrounded by crushed stone in a rectangular courtyard as shown. The crushed stone border has a uniform width x (in feet).

You have enough money to purchase tiles to cover 140 square feet. Solve the equation $140 = (20 - 24)(16 - 2x)$ to find the width of the border.



1.1 Quadratic Equations

Definition

If $P(x)$ is a polynomial of degree 2, then $P(x) = c$ is called a quadratic equation, where c is some constant.

$ax^2 + bx + c = 0$, $a \neq 0$ is a standard form of the quadratic equation in one variable x.

$(x + 1)^2 = 4$ and $x^2 + 3x = 18$ are examples of quadratic equation in one variable x but they are not in standard form.

Example 1: Write the following quadratic equations in standard form.

(i). $ax^2 + bx = c$ **(ii).** $(3x - 1)^2 = 4$ **(iii).** $(x + 1)^2 - (2x - 1)^2 = 2$

Solution:

i.
$$ax^2 + bx = c$$
$$ax^2 + bx - c = 0$$

ii.
$$(3x - 1)^2 = 4$$
$$9x^2 - 6x + 1 = 4$$
$$9x^2 - 6x + 1 - 4 = 0$$
$$9x^2 - 6x - 3 = 0$$
$$3(3x^2 - 2x - 1) = 0, \text{ as } 3 \neq 0$$
$$3x^2 - 2x - 1 = 0$$

iii.
$$(x + 1)^2 - (2x - 1)^2 = 2$$
$$(x^2 + 2x + 1) - (4x^2 - 4x + 1) = 2$$
$$x^2 + 2x + 1 - 4x^2 + 4x - 1 - 2 = 0$$
$$-3x^2 + 6x - 2 = 0$$
$$-1(3x^2 - 6x + 2) = 0$$
$$3x^2 - 6x + 2 = 0 \qquad \text{as } -1 \neq 0$$

1.2 Solution of Quadratic Equation

The values of variables for which an equation becomes a true sentence are called solutions or roots of the equation. i.e. if $P(a) = c$, then $x = a$ is root of the equation $P(x) = c$. The set of roots of an equation is called the solution set. A quadratic equation in one variable has two roots. These roots of the quadratic equation can be found in three ways.

i. Factorization Method

ii. Completing Square Method

iii. Using Quadratic Formula

8.2.1 Solution of Quadratic Equation by Factorization

i. To solve the quadratic equation by factorization first write it in standard form, then factorize the polynomial on the left hand side of the equation.

ii. Form two linear equations using the following rule:
" if the product of two quantities is zero then at least one of them is zero."
i.e. if $a \times b = 0$, then $a = 0$ or $b = 0$.

iii. Now solve linear equations and write solution set.
e.g. to solve $x^2 = 4$ by factorization we do the following steps:

$x^2 = 4$	
$x^2 - 4 = 0$	(standard form)
$(x)^2 - (2)^2 = 0$	
$(x - 2)(x + 2) = 0$	(factorization)
$x - 2 = 0$ or $x + 2 = 0$	(linear equations)
$x = 2$ or $x = -2$	(roots)
S.S. = $\{-2, 2\}$	(solution set)

Example 2: Solve the following equations by factorization.

(i). $x^2 + 7x + 12 = 0$ (ii). $(x + 1)^2 = 10 - (2x + 1)^2$

Solution:

i.
$$x^2 + 7x + 12 = 0$$
$$x^2 + 3x + 4x + 12 = 0$$
$$x(x + 3) + 4(x + 3) = 0$$
$$(x + 3)(x + 4) = 0$$
$$x + 3 = 0 \text{ or } x + 4 = 0$$
$$x = -3 \text{ or } x = -4$$
$$\text{S.S.} = \{-3, -4\}$$

ii.
$$(x + 1)^2 = 10 - (2x + 1)^2$$
$$(x + 1)^2 + (2x + 1)^2 - 10 = 0$$
$$x^2 + 2x + 1 + 4x^2 + 4x + 1 - 10 = 0$$
$$5x^2 + 6x - 8 = 0$$
$$5x^2 + 10x - 4x - 8 = 0$$
$$5x(x + 2) - 4(x + 2) = 0$$
$$(x + 2)(5x - 4) = 0$$
$$x + 2 = 0 \text{ or } 5x - 4 = 0$$
$$x = -2 \text{ or } 5x = 4$$
$$x = -2 \text{ or } x = \frac{4}{5}$$
$$\text{S.S.} = \{-2, \frac{4}{5}\}$$

1.2.2 Solution of Quadratic Equation by Completing Square Method

In this method, we do the following steps.

Step I: Write the equation in standard form.

Step II: Shift constant term to the right hand side of the equation.

Step III: To make the left hand side a complete square, add a suitable term to both sides of the equation. If the coefficient of x^2 is 1, then it is easy to complete the square. Just add the square of half the coefficient of x and get a complete square.

Step IV: Take the square root on both sides and form two linear equations.

Step V: Solve the linear equations and find roots.

Step VI: Write solution set.

Example 3:

Solve $3x^2 + 6x - 9 = 0$ by the completing square method.

Solution:

$$3x^2 + 6x - 9 = 0$$
$$3x^2 + 6x = 9$$
$$x^2 + 2x = 3 \quad \text{(dividing by 3)}$$
$$x^2 + 2x + 1 = 3 + 1 \quad \text{(adding 1 to both sides)}$$
$$(x + 1)^2 = 4$$
$$x + 1 = \pm\sqrt{4} \quad \text{(taking square root)}$$
$$x + 1 = \pm 2$$
$$x = \pm 2 - 1$$
$$x = 1 \quad \text{or} \quad x = -3$$

S.S. = $\{1, -3\}$

Describe and connect the error.

$x^2 - 14x = 11$

$x^2 - 14x + 49 = 11$

$(x - 7)^2 = 11$

$x - 7 = \pm\sqrt{11}$

$x = 7 \pm \sqrt{11}$

1.3 Quadratic Formula

Vocabulary

For an expression of the form $x^2 + bx$, you can add a constant c in the expression so that the expression $x^2 + bx + c$ is a perfect square trinomial. This process is called completing square.

1.3.1 Derivation of the Quadratic Formula

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x^2 + 2(x)\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Which is the required quadratic formula.

1.3.2 Solution of Quadratic Equations by Formula

To solve a quadratic equation by using the quadratic formula, we proceed as follows:

Step I: Write the equation in standard form.
Step II: Compare it with $ax^2 + bx + c = 0$, to get the values of a, b and c.
Step III: Write the quadratic formula for x.
Step IV: Put values of a, b and c, then simplify.
Step V: Write solution set.

Example 4:

Solve $(x + 1)^2 = 9$ by using quadratic formula.

Solution:

$(x + 1)^2 = 9$
$x^2 + 2x + 1 - 9 = 0$
$x^2 + 2x - 8 = 0$
Comparing with $ax^2 + bx + c = 0$, we get $a = 1$, $b = 2$ and $c = -8$

Now $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \frac{-2 \pm \sqrt{2^2-4(1)(-8)}}{2(1)} = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$x = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2}$$

$$x = \frac{-2+6}{2} \text{ or } x = \frac{-2-6}{2}$$

$$x = \frac{4}{2} = 2 \text{ or } x = -\frac{8}{2} = -4$$

$\therefore$ S.S. = $\{2, -4\}$

Check Point

The product of two consecutive negative integers is 210. Find the integer.

EXERCISE 1.1

1. Write the following quadratic equations in standard form.

i. $(x + 2)(x - 3) = 5$ **ii.** $(x - 5)^2 - (2x + 4)^2 = 7$ **iii.** $x = x(x - 1)$

2. Solve the following equations by factorization.

i. $(x - 1)(x - 4) = 0$ **ii.** $x^2 - 2x + 1 = 0$
iii. $x^2 - 7x - 8 = 0$ **iv.** $x^2 - 4x + 4 = (2x - 7)^2$
v. $(2x + \frac{7}{4})^2 = \frac{48x^2 + 529}{16}$

3. Solve the following equations by completing the square method.

i. $x^2 + 4x - 32 = 0$ **ii.** $x^2 + 8x = 0$ **iii.** $x^2 + 6x - 9 = 0$
iv. $3x^2 + 12x + 8 = 0$ **v.** $x^2 + x + 1 = 0$ **vi.** $4x^2 - 8x - 5 = 0$

4. Solve the following equations by quadratic formula.

i. $x^2 - 9 = 0$ ii. $2x^2 + 5x + 1 = 0$ iii. $x^2 - 23x - 24 = 0$

iv. $(x + 1)^2 = (2x - 1)^2$ v. $\frac{x+1}{2} - \frac{x(x+2)}{3} = 0$ vi. $(x - 2)(x - 6) = (2x + 1)(x + 1)$

1.4 Equations Reducible to Quadratic Form

Some equations are not quadratic, but to solve them we can reduce them to quadratic form. These equations are of many types, some of them are given below.

1.4.1 Equations of the Form $ax^4 + bx^2 + c = 0$

It is an equation of degree 4. It can be reduced to quadratic form by using the substitution $x^2 = y$. In this way the above equation becomes $ay^2 + by + c = 0$. Solve it by any of the three methods discussed earlier. Replace y with its substitute x^2. Find values of x by taking the square root and write the solution set.

Example 5: Solve $9x^4 - 8x^2 - 1 = 0$.

Solution:

$9x^4 - 8x^2 - 1 = 0$

Put $x^2 = y$ and $x^4 = y^2$

$9y^2 - 8y - 1 = 0$

$9y^2 - 9y + y - 1 = 0$

$9y(y - 1) + 1(y - 1) = 0$

$(y - 1)(9y + 1) = 0$

$y - 1 = 0$ or $9y + 1 = 0$

$y = 1$ or $y = -\frac{1}{9}$

$x^2 = 1$ or $x^2 = -\frac{1}{9}$

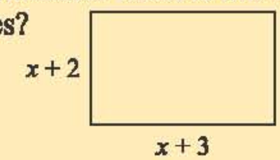
$x = \pm\sqrt{1}$ or $x = \pm\sqrt{-\frac{1}{9}}$

$x = \pm 1$ or $x = \pm\frac{1}{3}$

$\therefore$ S.S. $= \{\pm\frac{1}{3}, \pm 1\}$

Challenge

The rectangle has an area of 42 square centimeters. What is the value of length of sides?



1.4.2 Equations of the Form $a\text{P}(x) + \frac{b}{\text{P}(x)} = c$

To solve this type of equation, use a substitution $\text{P}(x) = y$. Multiply the equation by y and form a quadratic equation involving the variable y. Solve the new equation for y and replace y by $\text{P}(x)$. In this way, you will get two simple equations involving variable x. Solve these equations and write the solution set.

Example 6: Solve $3x + \frac{4}{x} = 7$.

Solution: $3x + \frac{4}{x} = 7$

$3x^2 + 4 = 7x$

$$3x^2 - 7x + 4 = 0$$
$$3x^2 - 3x - 4x + 4 = 0$$
$$3x(x-1) - 4(x-1) = 0$$
$$x - 1 = 0 \text{ or } 3x - 4 = 0$$
$$x = 1 \quad \text{or } x = \frac{4}{3}$$

$\therefore$ S.S. = $\{1, \frac{4}{3}\}$

Example 7: Solve $2(x^2+1) - \frac{2}{x^2+1} = 3$.

Solution: $2(x^2+1) - \frac{2}{x^2+1} = 3$

Put $x^2 + 1 = y$

So $2y - \frac{2}{y} = 3$

$$2y^2 - 2 = 3y$$
$$2y^2 - 3y - 2 = 0$$
$$2y^2 - 4y + y - 2 = 0$$
$$2y(y-2) + 1(y-2) = 0$$
$$(y-2)(2y+1) = 0$$
$$y - 2 = 0 \quad \text{or} \quad 2y + 1 = 0$$

Substituting the value of y, we get:

$$x^2 + 1 - 2 = 0 \quad \text{or} \quad 2(x^2+1) + 1 = 0$$
$$x^2 + 1 - 2 = 0 \quad \text{or} \quad 2x^2 + 2 + 1 = 0$$
$$x^2 - 1 = 0 \quad \text{or} \quad 2x^2 + 3 = 0$$
$$x^2 = 1 \quad \text{or} \quad x^2 = -\frac{3}{2}$$
$$x = \pm\sqrt{1} \text{ or } x = \pm\sqrt{-\frac{3}{2}}$$
$$x = \pm 1 \quad \text{or} \quad x = \pm\sqrt{\frac{3}{2}}\,i$$

1.4.3 Reciprocal Equations

If an equation is not affected by the replacement of a variable by its reciprocal, then it is called a reciprocal equation. The reciprocal of a root of such an equation is also its root.

Consider the equation $ax^4 + bx^3 + cx^2 + bx + a = 0$. $\rightarrow (i)$

$a\left(\frac{1}{x}\right)^4 + b\left(\frac{1}{x}\right)^3 + c\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + a = 0$ (x is replaced by $\frac{1}{x}$)

$\frac{a}{x^4} + \frac{b}{x^3} + \frac{c}{x^2} + \frac{b}{x} + a = 0$

$a + bx + cx^2 + bx^3 + ax^4 = 0$ (equation is multiplied by x^4)

$ax^4 + bx^3 + cx^2 + bx + a = 0.$ → ***(ii)***

Equations (i) and (ii) are same.

To solve the reciprocal equation (i), arrange it in the form

$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0.$

Put $y = x + \frac{1}{x}$ and find the value of $x^2 + \frac{1}{x^2} = y^2 - 2.$

In this way equation (i) is reduced to quadratic form. Solve this equation for y and then replace y with its substitute $x + \frac{1}{x}$. Finally find values of x and write solution set.

Example 8:

Solve the equation $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$

Solution:

$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

$6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$ (dividing by x^2)

$6(x^2 + \frac{1}{x^2}) - 35(x + \frac{1}{x}) + 62 = 0$ → ***(i)***

Let $x + \frac{1}{x} = y$

$(x + \frac{1}{x})^2 = y^2$

$x^2 + \frac{1}{x^2} + 2 = y^2$

$x^2 + \frac{1}{x^2} = y^2 - 2$

Put in equation (i)

$6(y^2 - 2) - 35y + 62 = 0$

$6y^2 - 12 - 35y + 62 = 0$

$6y^2 - 35y + 50 = 0$

$6y^2 - 15y - 20y + 50 = 0$

$3y(2y - 5) - 10(2y - 5) = 0$

$(2y - 5)(3y - 10) = 0$

$2y - 5 = 0$ or $3y - 10 = 0$

$2(x + \frac{1}{x}) - 5 = 0$ or $3(x + \frac{1}{x}) - 10 = 0$ (Substituting the value of y)

$2x + \frac{2}{x} - 5 = 0$	$3x + \frac{3}{x} - 10 = 0$
$2x^2 + 2 - 5x = 0$	$3x^2 + 3 - 10x = 0$
$2x^2 - 5x + 2 = 0$	$3x^2 - 10x + 3 = 0$
$2x^2 - 4x - x + 2 = 0$	$3x^2 - 9x - x + 3 = 0$

$2x(x - 2) - 1(x - 2) = 0$ or $3x(x - 3) - 1(x - 3) = 0$

$(x - 2)(2x - 1) = 0$ or $(x - 3)(3x - 1) = 0$

$x - 2 = 0$ or $2x - 1 = 0$ or $x - 3 = 0$ or $3x - 1 = 0$

$x = 2$ or $x = \frac{1}{2}$ or $x = 3$ or $x = \frac{1}{3}$

$\therefore$ S.S. $= \{2, \frac{1}{2}, 3, \frac{1}{3}\}$

1.4.4 Exponential Equations

An equation involving expressions of the form k^x is called an exponential equation where k is a constant. In exponential equations the exponents are variables. We will discuss only those exponential equations which can be reduced to quadratic form.

Exponential equations reducible to quadratic form are of the type:

$ak^{2x} + bk^x + c = 0, a \neq 0.$

We use the substitution $k^x = y$ to reduce the above equation to quadratic form. Then we find y and replace it with k^x. In this way, roots of the exponential equations are found.

Example 9:

Solve the following exponential equations

i. $2^{2x} - 2 \times 2^x + 1 = 0$

ii. $9^x - 3^{x+1} - 4 = 0$

Solution:

i. $2^{2x} - 2 \times 2^x + 1 = 0 \rightarrow (i)$

Put $2^x = y$ so $2^{2x} = y^2$

So equation (i) becomes

$y^2 - 2y + 1 = 0$

$(y - 1)^2 = 0$

$y - 1 = 0$

$y = 1$

Replace y by its substitute 2^x

$2^x = 1 = 2^0$

$x = 0$

$\therefore$ S.S. = {0}

ii. $9^x - 3^{x+1} - 4 = 0$

$3^{2x} - 3^x \times 3 - 4 = 0 \rightarrow (i)$

Put $3^x = y$ so $3^{2x} = y^2$

So equation (i) becomes

$y^2 - 3y - 4 = 0$

$y^2 - 4y + y - 4 = 0$

$y(y - 4) + 1(y - 4) = 0$

$(y - 4)(y + 1) = 0$

$y - 4 = 0$ or $y + 1 = 0$

$y = 4$ or $y = -1$

$3^x = 4$ or $3^x = -1$

Converting $3^x = 4$ into logarithmic form, we get

$x = \log_3 4$

Solution of $3^x = -1$ is not possible.

S.S. = $\{\log_3 4\}$

1.4.5 Equations of the Form $(x + a)(x + b)(x + c)(x + d) = k;\ a + b = c + d$

In this type, we multiply the pairs of factors after rearranging them in such a way that the variable terms of both products are the same or one of them is multiple of the other. Choose a substitute for the common terms and form a quadratic equation. Solve the new equation and then replace the substitute. Find the values of original variable involved in the given equation and write the solution set.

Example 10:

Solve the following equations by using suitable substitution.

$(x + 1)(x + 2)(x + 3)(x + 4) = 24$

Solution:

$(x + 1)(x + 2)(x + 3)(x + 4) = 24$

$(x + 1)(x + 4)\ (x + 2)(x + 3) = 24$

$(x^2 + 5x + 4)(x^2 + 5x + 6) = 24$

Put $x^2 + 5x = y$

$(y + 4)(y + 6) = 24$

$y^2 + 10y + 24 = 24$

$y^2 + 10y = 0$

$y(y + 10) = 0$

$y = 0$ or $y + 10 = 0$

For $y = 0$

$x^2 + 5x = 0$

$x(x + 5) = 0$

$x = 0$ or $x + 5 = 0$

$x = 0$ or $x = -5$

For $y + 10 = 0$

$x^2 + 5x + 10 = 0$

Here $a = 1$, $b = 5$ and $c = 10$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(10)}}{2(1)}$

$x = \frac{-5 \pm \sqrt{-15}}{2}$

$\therefore$ S.S. $= \{0, -5, \frac{-5 \pm \sqrt{-15}}{2}\}$

EXERCISE 1.2

1. Reduce the following equations to quadratic form using suitable substitution.

- **i.** $ax^4 + bx^2 + c = 0$
- **ii.** $9x^6 - 3x^3 + 7 = 0$
- **ii.** $3x + \frac{4}{6x-2} = -1$
- **iv.** $(x + 1)^2 + \frac{3}{(x+1)^2} = 4$
- **v.** $3(x^2 + \frac{1}{x^2}) + 8(x + \frac{1}{x}) + 11 = 0$
- **vi.** $3x^4 + 7x^3 + 5x^2 - 7x + 3 = 0$
- **vii.** $ak^{2x} + bk^x + c = 0$
- **viii.** $8 \times 4^x - 7 \times 2^x - 1 = 0$
- **ix.** $(x - 1)(x - 2)(x + 3)(x - 6) = 6$
- **x.** $(2x - 1)(2x - 7)(x - 3)(x - 1) = 8$

2. Solve the following equations by reducing them to quadratic form.

- **i.** $x^4 - 20x^2 + 64 = 0$
- **ii.** $x^4 + 16x^2 - 225 = 0$
- **iii.** $x^{\frac{2}{5}} + 5x^{\frac{1}{5}} + 6 = 0$
- **iv.** $3x^2 + \frac{4}{x^2} = 7$
- **v.** $5(x + 1) + \frac{3}{x+1} = 8$
- **vi.** $5x^2 + \frac{36}{5x^2+4} = 16$
- **vii.** $2x^4 - x^3 - 6x^2 - x + 2 = 0$
- **viii.** $12x^4 + 11x^3 - 146x^2 + 11x + 12 = 0$
- **ix.** $4\left(x^2 + \frac{1}{x^2}\right) - \left(x - \frac{1}{x}\right) - 11 = 0$
- **x.** $2^{2x} - 34 \times 2^x + 64 = 0$
- **xi.** $3^{2x} - 12 \times 3^x + 27 = 0$
- **xii.** $5^{2x} - 150 \times 5^x + 3125 = 0$
- **xiii.** $(x + 2)(x - 3)(x + 10)(x + 5) = -396$
- **xiv.** $x(x - 1)(x + 2)(x + 3) = 40$
- **xv.** $(x - 2)(x - 6)(x + 4)(x + 8) + 256 = 0$

3. Solve the following equations by factorization without using substitution.

- **i.** $2x^4 - 3x^2 + 1 = 0$
- **ii.** $8x^6 - 7x^3 - 1 = 0$ (Find only real roots.)
- **iii.** $x^2 + \frac{1}{x^2} = 2$
- **iv.** $4 \times 2^{2x} - 4 \times 2^x + 1 = 0$

1.5 Radical Equations

An equation involving at least one irrational expression of the type $\sqrt{P(x)}$ is called a radical equation. e.g. $\sqrt{x+3}+5=7$ is a radical equation. $\sqrt{3}x+5=7$ is not radical, because in this equation variable x is not present in the radical.

Before starting the solution of radical equations let us revise some basic rules and properties related to radicals.

i. $\sqrt{P(x)} = (P(x))^{\frac{1}{2}}$

ii. If $P(x) < 0$, then $\sqrt{P(x)} \notin R$

iii. $\sqrt{P(x)Q(x)} = \sqrt{P(x)}\,\sqrt{Q(x)}$

iv. $\sqrt{\frac{P(x)}{Q(x)}} = \frac{\sqrt{P(x)}}{\sqrt{Q(x)}}$, $Q(x) \neq 0$

v. $\left[\sqrt{P(x)}\right]^2 = P(x)$

vi. $\sqrt{[P(x)]^2} = \begin{cases} P(x) & \text{if } P(x) \geq 0 \\ -P(x) & \text{if } P(x) < 0 \end{cases}$

vii. It is necessary to check/verify the roots of the radical equation before writing the solution set.

viii. Sometimes roots of a radical equation do not satisfy it. Such roots are called *extraneous roots*.

ix. During verification of the root of a radical equation, it is not allowed to take square on both sides.

x. Extraneous roots are not included in the solution set.

xi. If all roots of a radical equation are extraneous then the solution set will be empty.

1.5.1 Radical Equations of type $\sqrt{ax+b} = cx+d$

To solve this type of radical equation, take a square on both sides and form a quadratic equation. Solve the quadratic equation and verify the answers.

Example 11: Solve the equation $\sqrt{3x-12} = x-4$ and check.

Solution:

$$\sqrt{3x-12} = x-4$$
$$\left(\sqrt{3x-12}\right)^2 = (x-4)^2$$
$$3x-12 = x^2-8x+16$$
$$3x-12-x^2+8x-16 = 0$$
$$-x^2+11x-28 = 0$$
$$x^2-11x+28 = 0$$
$$x^2-4x-7x+28 = 0$$
$$x(x-4)-7(x-4) = 0$$
$$(x-7)(x-4) = 0$$
$$x-7 = 0 \text{ or } x-4 = 0$$
$$x = 7 \text{ or } x = 4$$

Verification:

Put $x = 7$, $\sqrt{3x-12} = x-4$ $\sqrt{3(7)-12} = 7-4$ $\sqrt{9} = 3$ $3 = 3$ It is a true sentence, so $x = 7$ satisfies the equation.	Put $x = 4$, $\sqrt{3x-12} = x-4$ $\sqrt{3(4)-12} = 4-4$ $\sqrt{12-12} = 0$ $0 = 0$ It is a true sentence, so $x = 4$ satisfies the equation.

Therefore, S.S. = {7, 4}

1.5.2 Radical Equations of type $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

To solve this type of equation, take a square on both sides. You will get an equation having only one radical term. Rearrange the terms so that radical term is on the one side and all non radical terms are on the other side of the equation. Again square both sides and you will get a quadratic equation. Solve this quadratic equation and verify the roots. Finally write the solution set.

Example 12:

Solve the equation $\sqrt{x+1}+\sqrt{x-1}=\sqrt{3x-1}$ and check.

Solution:

$$\sqrt{x+1}+\sqrt{x-1}=\sqrt{3x-1}$$
$$\left(\sqrt{x+1}+\sqrt{x-1}\right)^2=\left(\sqrt{3x-1}\right)^2$$
$$x+1+x-1+2\sqrt{(x+1)(x-1)}=3x-1$$
$$2x+2\sqrt{x^2-1}=3x-1$$
$$2\sqrt{x^2-1}=3x-1-2x$$
$$2\sqrt{x^2-1}=x-1$$
$$\left(2\sqrt{x^2-1}\right)^2=(x-1)^2$$
$$4(x^2-1)=x^2-2x+1$$
$$4x^2-4-x^2+2x-1=0$$
$$3x^2+2x-5=0$$
$$3x^2-3x+5x-5=0$$
$$3x(x-1)+5(x-1)=0$$
$$(x-1)(3x+5)=0$$
$$x-1=0 \text{ or } 3x+5=0$$
$$x=1 \text{ or } x=-\frac{5}{3}$$

Therefore, S.S. = $\{1, -\frac{5}{3}\}$

Verification:

Put $x = 1$, $\sqrt{x+1}+\sqrt{x-1}=\sqrt{3x-1}$

$$\sqrt{1+1}+\sqrt{1-1}=\sqrt{3(1)-1}$$
$$\sqrt{2}+\sqrt{0}=\sqrt{3-1}$$
$$\sqrt{2}=\sqrt{2}$$

Which is a true sentence.

Put $x=-\frac{5}{3}$,

$$\sqrt{x+1}+\sqrt{x-1}=\sqrt{3x-1}$$
$$\sqrt{-\frac{5}{3}+1}+\sqrt{-\frac{5}{3}-1}=\sqrt{3(-\frac{5}{3})-1}$$
$$\sqrt{-\frac{2}{3}}+\sqrt{-\frac{8}{3}}=\sqrt{-\frac{18}{3}}$$
$$\sqrt{-\frac{2}{3}}+2\sqrt{-\frac{2}{3}}=\sqrt{-6}$$
$$3\sqrt{-\frac{2}{3}}=\sqrt{-6}$$
$$\sqrt{-6}=\sqrt{-6}$$

Which is a true sentence.

1.5.3 Radical Equations of type $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

To solve this type of equation, use a substitute $x^2+px=$ y. You will get an equation of the previous type involving variable y. Solve this equation and find y. Replace y with its substitute and find the values of x. Verify the roots and write the solution set.

Example 13:

Solve the equation: $\sqrt{x^2+10x+4}+\sqrt{x^2+10x+3}=2$

Solution: $\sqrt{x^2+10x+4}+\sqrt{x^2+10x+3}=2$

Put $x^2+10x=y$

$\sqrt{y+4}+\sqrt{y+3}=2$

$\left(\sqrt{y+4}+\sqrt{y+3}\right)^2=2^2$

$y+4+y+3+2\sqrt{y+4}\sqrt{y+3}=4$

$2\sqrt{(y+4)(y+3)}=4-2y-7$

$2\sqrt{y^2+7y+12}=-2y-3$

$\left(2\sqrt{y^2+7y+12}\right)^2=(-2y-3)^2$

$4(y^2+7y+12)=4y^2+12y+9$

$4y^2+28y+48-4y^2-12y-9=0$

$16y+39=0$, put $y=x^2+10x$

$16(x^2+10x)+39=0$

$16x^2+160x+39=0$

$16x^2+4x+156x+39=0$

$4x(4x+1)+39(4x+1)=0$

$(4x+39)(4x+1)=0$

$4x+39=0$ or $4x+1=0$

$x=-\frac{39}{4}$ or $x=-\frac{1}{4}$

Alternate Method

Solution: Let $a=\sqrt{x^2+10x+4}$

and $b=\sqrt{x^2+10x+3}$

So $a+b=2 \rightarrow$ ***(i)***

$a^2-b^2=(x^2+10x+4)-(x^2+10x+3)$

$a^2-b^2=1$

$(a+b)(a-b)=1$

$2(a-b)=1$ from ***(i)*** $a+b=2$

$2a-2b=1 \rightarrow$ ***(ii)***

Multiplying equation(i) by 2,

then adding in equation (ii) we get

$4a=5$

$16a^2=25$

$16(x^2+10x+4)=25$

$16x^2+160x+64-25=0$

$16x^2+160x+39=0$

$16x^2+4x+156x+39=0$

$4x(4x+1)+39(4x+1)=0$

Challenge

Can you solve this equation?

$\sqrt{x^2+7x+10}+\sqrt{x^2+7x-25}=7$

If yes then what is your opinion about the verification of solution?

Challenge

In example 13, if we put

$x^2+10x+3=y$

then we get $\sqrt{y+1}+\sqrt{y}=2$

$\left(\sqrt{y+1}+\sqrt{y}\right)^2=2^2$

and so on….

Can you solve this problem?

If yes, then solve questions 11-13 of exercise 1.3 by using this technique.

Verification:

Put $x=-\frac{39}{4}$

$\sqrt{\left(\frac{-39}{4}\right)^2+10\left(\frac{-39}{4}\right)+4}+\sqrt{\left(\frac{-39}{4}\right)^2+10\left(\frac{-39}{4}\right)+3}=2$

$\sqrt{\frac{1521}{16}-\frac{390}{4}+4}+\sqrt{\frac{1521}{16}-\frac{390}{4}+3}=2$

$\sqrt{\frac{1521-1560+64}{16}}+\sqrt{\frac{1521-1560+48}{16}}=2$

$\sqrt{\frac{25}{16}}+\sqrt{\frac{9}{16}}=2$

$\frac{5}{4}+\frac{3}{4}=2$

$2=2$

Which is a true sentence.

Put $x=-\frac{1}{4}$

$\sqrt{\left(\frac{-1}{4}\right)^2+10\left(\frac{-1}{4}\right)+4}+\sqrt{\left(\frac{-1}{4}\right)^2+10\left(\frac{-1}{4}\right)+3}=2$

$\sqrt{\frac{1}{16}-\frac{10}{4}+4}+\sqrt{\frac{1}{16}-\frac{10}{4}+3}=2$

$\sqrt{\frac{1-40+64}{16}}+\sqrt{\frac{1-40+48}{16}}=2$

$\sqrt{\frac{25}{16}}+\sqrt{\frac{9}{16}}=2$

$\frac{5}{4}+\frac{3}{4}=2$

$2=2$

Which is a true sentence.

$$(4x + 1)(4x + 39) = 0$$
$$4x + 1 = 0 \quad \text{or} \quad 4x + 39 = 0$$
$$4x = -1 \quad \text{or} \quad 4x = -39$$
$$x = -\frac{1}{4} \quad \text{or} \quad x = -\frac{39}{4}$$

Therefore, S.S. = $\{-\frac{39}{4}, -\frac{1}{4}\}$

EXERCISE 1.3

Solve the following radical equations and check your answer.

1. $\sqrt{3x+6} = x + 2$
2. $3\sqrt{4-5x} = 10x - 8$
3. $2\sqrt{3x+4} - 4 + 3x = 0$
4. $5 - 4\sqrt{7x-1} = 9 - 28x$
5. $2x - 7 = \sqrt{3x-11}$
6. $\sqrt{3x+4} - \sqrt{2x+1} = \sqrt{x+1}$
7. $\sqrt{x+2} - \sqrt{x+1} = 2$
8. $\sqrt{x+1} - \sqrt{x-1} = \sqrt{3x-1}$
9. $\sqrt{2x+1} + \sqrt{3x-1} = \sqrt{9x+2}$
10. $\sqrt{x-1} + \sqrt{2x-1} = \sqrt{5x-4}$
11. $\sqrt{x^2+x+9} + \sqrt{x^2+x+4} = 5$
12. $\sqrt{x^2+x+5} - \sqrt{x^2+x+26} = 7$
13. $\sqrt{x^2+4x+13} + \sqrt{x^2+4x+5} = 2$

KEY POINTS

- $ax^2 + bx + c = 0, a \neq 0$ is standard form of quadratic equation in one variable x.
- There are three methods for solving a quadratic equation.
  i. Factorization Method **ii.** Completing Square Method **iii.** The Quadratic Formula
- Some equations are not quadratic but they can be reduced in quadratic form.
- An equation involving at least one irrational expression of the type $\sqrt{P(x)}$ is called a radical equation.
- Sometimes roots of a radical equation do not satisfy it. Such roots are called extraneous roots.

REVIEW EXERCISE 1

1. Encircle the correct option.

(i) Which of the following is a quadratic equation?

(a) $ax + b = c$ **(b)** $ax^2 + bx + c$

(c) $ax^2 + bx + c = 0, a \neq 0$ **(d)** $ax^2 + bx + c = 0, a = 0$

(ii) How many roots of $(x-3)(x-2)=6$ exist?

(a) no **(b)** 0 **(c)** 1 **(d)** 2

(iii) What should be added to $x^2 + x$ to make it a complete square?

(a) 1 **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{2}$ **(d)** 4

(iv) Solution set of $x^2 - 4 = 0$ is

(a) $\{0, 4\}$ **(b)** $\{2, -2\}$ **(c)** $\{4, -4\}$ **(d)** $\{\ \}$

(v) Roots of the equation $(x-1)^2 = 9$ are

(a) $-2, 4$ **(b)** $2, 4$ **(c)** $-4, 2$ **(d)** $-2, -4$

(vi) Solution set of $2^{2x} - 2^{x+1} + 1 = 0$?

(a) $\{0\}$ **(b)** $\{1\}$ **(c)** $\{0, 1\}$ **(d)** $\{0, -1\}$

(vii) Solution set of $x + \frac{1}{x} = 2$ is

(a) $\{0\}$ **(b)** $\{-1\}$ **(c)** $\{-1, 1\}$ **(d)** $\{1\}$

(viii) If $a + c = b$, then which of the following are the root of $ax^2 + bx + c = 0$?

(a) $-1, -\frac{c}{a}$ **(b)** $-1, -\frac{a}{c}$ **(c)** $1, -\frac{c}{a}$ **(d)** $-1, \frac{c}{a}$

(ix) Which of the following is a radical equation?

(a) $\sqrt{2}x + 3 = 0$ **(b)** $\sqrt{2}x + \sqrt{3} = 0$ **(c)** $2x + \sqrt{3} = 0$ **(d)** $\sqrt{2x} + 3 = 0$

(x) The roots which do not satisfy the equation are called:

(a) complex roots **(b)** imaginary roots

(c) extraneous roots **(d)** equal roots

(xi) Which of the following is a reciprocal equation?

(a) $x^2 + 2x + 2 = 0$ **(b)** $x^4 + x^3 + x^2 + x + 1 = 0$

(c) $\sqrt{2x+3} = 0$ **(d)** $x^4 + 2x^3 + x^2 + 4x = 0$

(xii) 2 and -3 are roots of

(a) $(x-2)(x-3) = 0$ **(b)** $(x+2)(x+3) = 0$

(c) $(x-2)(x+3) = 0$ **(d)** $(x+2)(x-3) = 0$

2. Find all roots of $8x^6 - 7x^3 - 1 = 0$

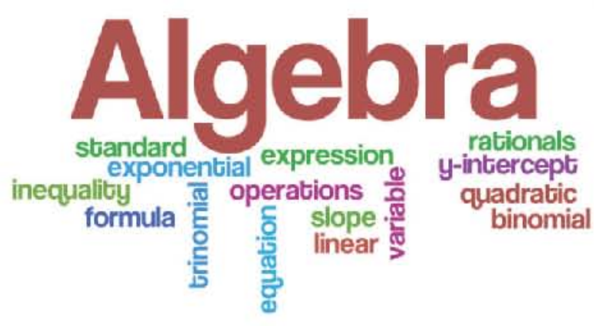
3. Solve $\sqrt{x^2 + 6x + 45} - \sqrt{x^2 + 6x + 10} = 5$

UNIT 02 Theory of Quadratic Equations

In this unit the students will be able to:

- Know nature of roots of a quadratic equation.
- Know about cube roots of unity and their properties.
- Find relation between roots and coefficients of a quadratic equation.
- Define and evaluate symmetric functions of roots of a quadratic equation.
- Form a quadratic equation when its roots are given.
- Use the method of synthetic division.
- Solve simultaneous equations.

Algebra is your trusted tool that helps you carry out of varies activities daily importance. There is hardly any line of work that does not employ the concept of algebra. So, next, we will look at the variables and equation, you surely will not be wondered about why you need to leave there in a nutshell. Algebra and equations prepare you for hardly all aspects of life and stays with you right from your infanthood to your adulthood.



2.1 Nature of Roots of a Quadratic Equation

You have learnt several ways to solve quadratic equations. Each has its own limitations. The question arises, "is there any formula that will work for all types of quadratic equations"? The answer is "yes" and that formula is called the "Quadratic Formula". We have already derived the quadratic formula by solving the general form of quadratic equation for x.

We know that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$, are given by this formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To define discriminant, first we solve the following examples by using the quadratic formula:

Example 1: $x^2 - 3x - 28 = 0$

Solution: We have $a = 1$, $b = -3$ and $c = -28$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-28)}}{2(1)} = \frac{3 \pm \sqrt{121}}{2}$$

$$x = \frac{3 \pm 11}{2} = 7, -4$$

The roots of equation are rational and unequal.

Example 2: $x^2 - 8x + 16 = 0$

Solution: We have $a = 1$, $b = -8$ and $c = 16$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)} = \frac{8 \pm 0}{2}$$

$$x = \frac{8+0}{2}, x = \frac{8-0}{2}$$

$$x = 4, x = 4$$

The roots of equation are rational and equal.

Example 3: $3y^2 - 5y + 9 = 0$

Solution: We have $a = 3$, $b = -5$ and $c = 9$

$$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(9)}}{2(3)} = \frac{5 \pm \sqrt{-83}}{6}$$

Since radical contains a negative value, therefore the roots of equation are imaginary.

Example 4: $3t^2 - 6t + 2 = 0$

Solution: We have $a = 3$, $b = -6$ and $c = 2$

$$t = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} = \frac{6 \pm \sqrt{12}}{6}$$

$$t = \frac{3 \pm \sqrt{3}}{3}$$

The roots are irrational and unequal.

These four examples demonstrate a pattern that is useful in determining the nature of the roots of a quadratic equation. In the quadratic formula, the expression under the sign '$b^2 - 4ac$', is called the discriminant. The discriminant tells the nature of roots of a quadratic equation.

Equation	Value of the discriminant	Roots	Nature of Roots
$x^2 - 3x - 28 = 0$ (Exp. 1)	$b^2 - 4ac = 121$	$7, -4$	Roots are rational and unequal.
$x^2 - 8x + 16 = 0$ (Exp. 2)	$b^2 - 4ac = 0$	$4, 4$	Roots are rational and equal.
$3y^2 - 5y + 9 = 0$ (Exp. 3)	$b^2 - 4ac = -83$	$\frac{5 \pm i\sqrt{83}}{6}$	Roots are imaginary/complex.
$3t^2 - 6t + 2 = 0$ (Exp. 4)	$b^2 - 4ac = 24$	$\frac{3 \pm \sqrt{3}}{3}$	Roots are irrational and unequal.

The above chart shows that if the value of the discriminant is a perfect square or 0, the roots are real and rational. Other positive discriminant will yield irrational roots. A negative discriminant means roots will be imaginary/complex.

Example 5:
Find the value of discriminant and describe the nature of roots.
Solution:

a. $2x^2 + x - 3 = 0$ $a = 2, b = 1, c = -3$ Disc $= b^2 - 4ac$ $= (1)^2 - 4(2)(-3)$ $= 25$ The value of discriminant is positive and a perfect square. So, the given equation has two real roots and they are rational and unequal.	b. $x^2 + 8 = 0$ $a = 1, b = 0, c = 8$ Disc $= b^2 - 4ac$ $= (0)^2 - 4(1)(8)$ $= -32$ The value of discriminant is negative. So, the given equation has two imaginary roots.

Check Point

Verify the nature of roots of $9x^2 - 12x + 4 = 0$, by solving it.

Example 6: Find the value of m, when $x^2 - 3x + m = 0$ has equal roots.

Solution: $x^2 + 3x + m = 0$

Here $a = 1$, $b = 3$, $c = m$

It is given that equation has equal roots (same root) so the discriminant is zero.

i.e. $\text{Disc} = b^2 - 4ac = 0$

$(3)^2 - 4(1)(m) = 0$

$9 - 4m = 0$

$4m = 9 \quad \text{or} \quad m = \frac{9}{4}$

EXERCISE 2.1

1. Find the discriminant of the following quadratic equations.
   (i) $x^2 + 6x - 27 = 0$ (ii) $x^2 - x - 12 = 0$
   (iii) $8x^2 + 2x + 1 = 0$ (iv) $12x^2 - 11x - 15 = 0$
2. Discuss the nature of roots of the following quadratic equations.
   (i) $x^2 - 2x - 15 = 0$ (ii) $x^2 + 3x - 4 = 0$ (iii) $12x^2 + x - 20 = 0$
   (iv) $x^2 + 2x + 8 = 0$ (v) $x^2 + 3x - 9 = 0$
3. For what value of k, $9k^2 - kx + 16 = 0$ is a perfect square?
4. If roots of $x^2 + kx + 9 = 0$ are equal, find k?
5. Show that the roots of $2x^2 + (mx - 1)^2 = 3$, are equal if $3m^2 + 4 = 0$.
6. Find the value of "m" when roots of the following quadratic equations are equal.
   (i) $x^2 - 6x + m = 0$ (ii) $m^2 x^2 + (2m + 1)x + 1 = 0$
   (iii) $(m + 3)x^2 + (m + 1)x + m + 1 = 0$
7. Show that the roots of the equation $(a^2 + b^2)x^2 + 2(ac + bd)x + c^2 + d^2 = 0$ are imaginary. Moreover, it shows repeated roots if $ad = bc$.
8. Show that the roots of the equation $(ax + c)^2 = 4bx$ will be equal, if $b = ac$.
9. Show that the roots of the following equations are real.
   (i) $mx^2 - 2mx + m - 1 = 0$ (ii) $bx^2 + ax + a - b = 0$
10. Show that the roots of the following equation is real.
    $(a + b)x^2 - ax - b = 0$

2.2 Cube Roots of Unity and their Properties

Let "x" be the cube root of unity, then

$$x = \sqrt[3]{1} = 1^{\frac{1}{3}}$$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x^3 - 1 = 0$$

$$\Rightarrow (x - 1)(x^2 + x + 1) = 0$$

$$\Rightarrow x - 1 = 0 \quad \text{or} \quad x^2 + x + 1 = 0$$

For $x - 1 = 0$, we have $x = 1$

For $x^2 + x + 1 = 0$, we have $a = 1, b = 1, c = 1$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{(-1)^2-4\times1\times1}}{2\times1}=\frac{-1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}$$

$$\Rightarrow x=\frac{-1+i\sqrt{3}}{2} \text{ and } x=\frac{-1-i\sqrt{3}}{2}$$

Thus, the three cube roots of unity are:

$$1,\ \frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}$$

We know that the numbers containing "i" are called complex number.

So, $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$ are the complex cube roots of unity while '1' is the real root.

2.2.1 Properties of Cube Roots of Unity

(1) Each complex cube root of unity is square of the other.

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^2=\frac{(-1)^2+(i\sqrt{3})^2+2(-1)(i\sqrt{3})}{4}$$

$$=\frac{1+3i^2-2i\sqrt{3}}{4}=\frac{1-3-2i\sqrt{3}}{4}$$

$$=\frac{-2-2i\sqrt{3}}{4}=2\left(\frac{-1-i\sqrt{3}}{4}\right)$$

$$=\frac{-1-i\sqrt{3}}{2}$$

Similarly

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^2=\left(-\frac{1+i\sqrt{3}}{2}\right)^2$$

$$=\frac{(1)^2+(i\sqrt{3})^2+2(1)i\sqrt{3}}{4}$$

$$=\frac{-2+2i\sqrt{3}}{4}=\frac{-1+i\sqrt{3}}{2}$$

Hence, each complex cube root of unity is square of the other.

History a Mystery

Until the sixteenth century, mathematicians were puzzled by square roots of negative numbers. As you know, some expressions have irrational solutions.

For example, the solution to $x^2-5=0$ are $\sqrt{5}$ and $-\sqrt{5}$.

But the equation $x^2=-1$ has no solution in the real numbers.

This is because the square of a real number is nonnegative. However, in 1545, the Italian mathematician Girolamo Cardano published Ars Magna in which he began working with what the great mathematician Rene Descartes later called imaginary numbers.

If $\frac{-1+i\sqrt{3}}{2}=\omega$, then $\frac{-1-i\sqrt{3}}{2}=\omega^2$ and if $\frac{-1-i\sqrt{3}}{2}=\omega$, then $\frac{-1+i\sqrt{3}}{2}=\omega^2$

Thus, three roots of unity are:

$$1, \quad \omega, \quad \omega^2$$

(2) Sum of all the three cube roots of unity is zero.

i.e. $1+\omega+\omega^2=0$

Sum of all three cube roots

$$= 1+\omega+\omega^2 = 1+\left(\frac{-1+i\sqrt{3}}{2}\right)+\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= \frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}=\frac{0}{2}=0$$

$\therefore$ We have, $1+\omega+\omega^2=0$.

(3) The product of all the three cube roots of unity is 1.

$$1.\omega.\omega^2 = 1.\left(\frac{-1+i\sqrt{3}}{2}\right).\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=\frac{(-1)^2-(i\sqrt{3})^2}{4}$$

$$=\frac{1-(-3)}{4}$$

$$\omega^3=\frac{1+3}{4}=1$$

Hence, the product of the cube root of unity is 1.

(4) For any $n\in Z$, ω^n is equivalent to one of the cube roots of unity.

Consider

$$\omega^4=\omega^3.\omega=1.\omega=\omega$$

$$\omega^5=\omega^3.\omega^2=1.\omega^2=\omega^2$$

$$\omega^6=(\omega^3)^2=(1)^2=1$$

$$\omega^8=\omega^6.\omega^2=(\omega^3)^2.\omega^2=1.\omega^2$$

$$\omega^{15}=(\omega^3)^5=1$$

Similarly

$$\omega^{-1}=\omega^{-3}.\omega^2=(\omega^3)^{-1}.\omega^2=(1)^{-1}\omega^2=\omega^2$$

$$\omega^{-2}=\omega^{-3}\omega^1=(\omega^3)^{-1}.\omega=(1)^{-1}.\omega=\omega$$

$$\omega^{-12}=(\omega^3)^{-4}=(1)^{-4}=1$$

Example 7:

Find three cube roots of 216 and show that sum of roots is zero.

Solution: Let x be the cube root of 216, then

$$x = (216)^{1/3}$$
$$\Rightarrow x^3 = 216$$
$$\Rightarrow x^3 - 216 = 0$$
$$\Rightarrow x^3 - 6^3 = 0$$
$$\Rightarrow (x-6)(x^2+6x+36) = 0$$
$$\Rightarrow x-6=0; \quad x^2+6x+36=0$$
$$x=6; \quad x=\frac{-6\pm\sqrt{36-4\times1\times36}}{2}$$
$$x=\frac{-6\pm\sqrt{-108}}{2}=\frac{-6\pm6\sqrt{-3}}{2}$$
$$x=6\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$
$$x=6\left(\frac{-1+\sqrt{-3}}{2}\right), \quad x=6\left(\frac{-1-\sqrt{-3}}{2}\right)$$
$$x=6\omega, \quad x=6\omega^2$$

> **Challenge**
>
> The volume of a small tin is $6x^3 - 4x^2 - 16x$ cm^3, where x represents the width of the tin. If the height of the tin is four centimeters more than three times the width. What is the length of the tin?

So, the cube root of 216 are: $6, \ 6\omega, \ 6\omega^2$

Note: Sum of three cube roots of 6 is zero.

i.e. $6+6\omega+6\omega^2 = 6(1+\omega+\omega^2) = 6(0) = 0$

Example 8: Prove that

$$(x^3+y^3) = (x+y)(x+\omega y)(x+\omega^2 y)$$

Proof:

$$\begin{aligned}
\text{R.H.S} &= (x+y)(x+\omega y)(x+\omega^2 y)\\
&= (x+y)(x^2+x\omega^2 y+x\omega y+\omega^3 y^2)\\
&= (x+y)(x^2+xy(\omega+\omega^2)+\omega^3 y^2)\\
&= (x+y)(x^2+xy(-1)+y^2)\\
&= (x+y)(x^2-xy+y^2)\\
&= x^3+y^3\\
&= \text{L.H.S}
\end{aligned}$$

EXERCISE 2.2

1. Find the three cube roots of
   (i) 8 (ii) − 8 (iii) 27 (iv) 64

2. Evaluate

(*i*) $\omega^{28}+\omega^{29}+1$ (*ii*) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

(*iii*) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^7$

3. Show that:

(i) $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

(ii) $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

(iii) $(1+3\omega+\omega^2)^6=64$ (iv) $1+\omega^{40}+\omega^{50}=0$ (v) $\omega^{48}+\omega^{49}+\omega^{50}=0$

2.3 Roots and Coefficients of the Quadratic Equation

Let α,β be the roots of $ax^2+bx+c=0$, then using the quadratic formula:

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}\ ,\ \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Sum of roots $=\alpha+\beta=\dfrac{-b+\sqrt{b^2-4ac}}{2a}+\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=\frac{-b}{a}$$

and product of roots $=\alpha\beta=\left(\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right)$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$=\frac{4ac}{4a^2}=\frac{c}{a}$$

$\therefore$ Sum of the roots $=$ S $=-\dfrac{b}{a}=-\dfrac{\text{coefficient of } x}{\text{coeficient of } x^2}$

and product of roots $=$ P $=\dfrac{c}{a}=\dfrac{\text{contant term}}{\text{coefficient of } x^2}$

Example 9: Write down the sum and product of the roots of quadratic equation:

(*i*) $x^2-x+2=0$ (*ii*) $x^2-7x+5=0$

(*iii*) $3x^2-11x-4=0$ (*iv*) $6s^2+2s+3=0$

Solution:

(i) $x^2 - x + 2 = 0$ Here $a = 1$, $b = -1$, $c = 2$ Sum of the roots = $\alpha + \beta$ $= \frac{-b}{a} = \frac{-(-1)}{1} = 1$ Product of roots = $\alpha\beta$ $= \frac{c}{a} = \frac{2}{1} = 2$	(ii) $3x^2 - 11x - 4 = 0$ $a = 3$, $b = -11$, $c = -4$ Sum of roots = $\alpha + \beta$ $= \frac{-b}{a} = \frac{-(-11)}{3} = \frac{11}{3}$ Product of roots = $\alpha\beta$ $= \frac{c}{a} = \frac{-4}{3}$
(iii) $x^2 - 7x + 5 = 0$ $a = 1$, $b = -7$, $c = 5$ $\alpha + \beta = \frac{-b}{a} = \frac{-(-7)}{1} = 7$ $\alpha\beta = \frac{c}{a} = \frac{5}{1} = 5$	(iv) $6s^2 + 2s + 3 = 0$ $a = 6$, $b = 2$, $c = 3$ $\alpha + \beta = \frac{-b}{a} = \frac{-(2)}{6} = \frac{-1}{3}$ $\alpha\beta = \frac{c}{a} = \frac{3}{6} = \frac{1}{2}$

Example 10: If α, β are the roots of $x^2 - 8x + 20 = 0$. Find the values of

(i) $\alpha^2 + \beta^2$ (ii) $(\alpha - \beta)^2$ (iii) $\frac{1}{\alpha+1} + \frac{1}{\beta+1}$

Solution: Here $a = 1$, $b = -8$, $c = 20$

Since α, β are the roots of $x^2 - 8x + 20 = 0$,

$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-8)}{1} = 8$ and $\alpha\beta = \frac{c}{a} = \frac{20}{1} = 20$

(i) Now $\alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$
$$= (8)^2 - 2(20)$$
$$= 64 - 40 = 24$$

(ii) $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

$$= \alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta$$
$$= (\alpha + \beta)^2 - 4\alpha\beta$$
$$= (8)^2 - 4(20)$$
$$= 64 - 80 = -16$$

(iii) $\frac{1}{\alpha+1} + \frac{1}{\beta+1} = \frac{\beta+1+\alpha+1}{(\beta+1)(\alpha+1)}$

$$= \frac{\alpha+\beta+2}{\alpha\beta + (\alpha+\beta) + 1}$$
$$= \frac{8+2}{20+8+1} = \frac{10}{29}$$

2.4 Formation of an Equation whose Roots are Given

If α, β are the roots of required quadratic equation, then

Let $x = \alpha \quad x = \beta$

$\Rightarrow \quad x - \alpha = 0, \quad x - \beta = 0$

$(x-\alpha)(x-\beta) = 0$

$x^2 - (\alpha+\beta)x + \alpha\beta = 0,$

$\alpha + \beta = \text{Sum of roots} = S, \quad \alpha\beta = \text{Product of roots} = P$

Then, $\quad x^2 - Sx + P = 0 \ldots\ldots\ldots (i)$

Above formula (i) is used to form a quadratic equation when its roots are given.
Let us solve some examples to understand the concept.

Example 11: Form a quadratic equation with roots 5 and 6.

Solution:

Since 5 and 6 are roots of the quadratic equation then

S = sum of roots = 5 + 6 = 11

P = product of the roots = 5(6) = 30

The required equation is

$x^2 - Sx + P = 0 \ldots\ldots\ldots (i)$

Substituting these values of S and P in (*i*), we have

$x^2 - 11x + 30 = 0$

Example 12:

If α, β are the roots of $ax^2 + bx + c = 0$. Form the equation whose roots are double to the roots of the given equation.

Solution:

If α, β are the roots of $ax^2 + bx + c = 0$ then

$\alpha + \beta = \frac{-b}{a} \quad , \quad \alpha\beta = \frac{c}{a}$

The new roots are 2α and 2β

Sum of new roots = $S = 2\alpha + 2\beta$

$= 2(\alpha+\beta) = 2(\frac{-b}{a}) \quad = \frac{-2b}{a}$

Product of new roots $= P = (2\alpha)(2\beta) = 4\alpha\beta$

$= 4\left(\frac{c}{a}\right) = \frac{4c}{a}$

Now substituting the values of S and P in $x^2 - Sx + P = 0$, we get

$x^2 - \left(\frac{-2b}{a}\right)x + \frac{4c}{a} = 0$

or $\quad ax^2 + 2bx + 4c = 0$

EXERCISE 2.3

1. Find the sum and product of the roots of the quadratic equation.

(i) $x^2-5x+2=0$ (ii) $-4x^2-6x-2=0$

(iii) $5x^2-2x+2=0$ (iv) $-4x^2-8x-9=0$

2. if $\alpha,\ \beta$ are the roots of $3x^2-2x+4=0,$ find the value of

(i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ (ii) $\alpha^2+\beta^2$ (iii) $2\alpha+2\beta+4$ (iv) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

(v) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$ (vi) $\alpha\beta^2+\alpha^2\beta$ (vii) $\alpha^3\beta+\alpha\beta^3$ (viii) $(\alpha-3)(\beta-3)$

3. if $\alpha,\ \beta$ are the roots of $7x^2+10x+7=0,$ form the equations whose roots are:

(i) $\alpha^2,\ \beta^2$ (ii) $\dfrac{1}{\alpha},\ \dfrac{1}{\beta}$ (iii) $\alpha^3\beta,\ \alpha\beta^3$ (iv) $\alpha-\dfrac{1}{\alpha},\ \beta-\dfrac{1}{\beta}$

(v) $2\alpha+1,\ 2\beta+1$ (vi) $\dfrac{\alpha}{\beta},\ \dfrac{\beta}{\alpha}$ (vii) $\alpha+\beta,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}$ (viii) $\alpha^2+1,\ \beta^2+1$

(ix) $\alpha^2+\beta,\ \alpha+\beta^2$

4. If α,β are the roots of $x^2+6x+3=0,$ form the equation whose roots are $(\alpha+\beta)^2,\ (\alpha-\beta)^2$

5. If α,β are the roots of $2x^2+6x-3=0,$ form the equation whose roots are $\alpha-\dfrac{3}{\beta^2},\ \beta-\dfrac{3}{\alpha^2}$

6. Find k if α and $\alpha-5$ are the roots of $x^2-3kx+5=0.$

2.3 Synthetic Division

Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial. This is a nice shortest method for long division of polynomial by a linear polynomial.

Outlines of the Method

i) Write down the coefficient of the dividend p(x) from left to right in decreasing order of powers to x. Insert '0' for any missing terms.

ii) To the left of the first line, write 'a' of the divisor $(x-a)$.

iii) Use the following patterns to write the second and third lines.

Vertical Pattern (↓) Add terms

Diagonal Pattern (↗) multiply by 'a'.

The last number of third line is 'remainder'.

Example 13: Use synthetic division to find the quotient and remainder when the polynomial x^4-10x^2-2x+4 is divided by $x+3$.

Solution: Let dividend = $p(x) = x^4-10x^2-2x+4$

$= x^4-0x^3-10x^2-2x+4$

and divisor = $x-a = x+3$

$\Rightarrow a = -3$

Now using method of synthetic division:

−3	1	0	−10	−2	4
	↓	−3	9	3	−3
	1	−3	−1	1	1 → Remainder

$\therefore$ Quotient $= x^3-3x^2-x+1$ and Remainder $= 1$

Example 14: Find quotient and remainder when the polynomial x^4-x^2+15 is divided by $x+1$.

Solution: Let $p(x) = x^4-x^2+15 = x^4+0x^3-x^2+0x+15$

Here $x-a = x+1$

$\Rightarrow a = -1$

Write the coefficients of dividend in a row as shown below with divisor −1 at the left end.

−1	1	0	−1	0	15
	↓	−1	1	0	0
	1	−1	0	0	15 → Remainder

$\therefore$ Quotient $= x^3-x^2+0x+0 = x^3-x^2$

Remainder $= 15$

Example 15:

Use synthetic division to find the value of unknown 'a' in the polynomial if the zero of polynomial $p(x) = 3x^2+4x-7a$ is 1.

Solution: Here '1' is the zero of polynomial $p(x)=3x^2+4x-7a$.

Using synthetic division, we have

1	3	4	$-7a$
		3	7
	3	7	$7-7a$ → Remainder

As '1' is the zero of polynomial p(x), therefore remainder will be zero.

i.e. $7-7a=0$

$7a=7$ or $a=1$

Example 16:

If $x+1$ and $x-2$ are the factors of x^3+px^2+qx+2, find the values of p and q.

Solution: $x+1=0 \Rightarrow x=-1$

$x-2=0 \Rightarrow x=2$

Let $f(x)=x^3+px^2+qx+2$

Using synthetic division, we have

-1	1	p	q	2
		-1	$-p+1$	$p-q-1$
2	1	$p-1$	$-p+q+1$	$p-q+1$ → Remainder
		2	$2p+2$	
	1	$p+1$	$p+q+3$ → Remainder	

As, $x+1$ and $x-2$ are factors of x^3+px^2+qx+2.

Therefore, remainders must be zero.

i.e. $p-q+1=0$.......(*i*) and $p+q+3=0$........(*ii*)

Adding (i) and (ii), we get

$2p+4=0$ or $2p=-4$

$\Rightarrow p=-2$

Substituting the value of p in equation (ii), we get

$-2+q+3=0$ or $q+1=0$

$\Rightarrow q=-1$

Example 17:

Using synthetic division, solve the equation $x^3+2x^2-9x-18=0$ when '– 2' is the root of equation.

Solution: As '– 2' is the root of equation $x^3+2x^2-9x-18=0$

Therefore, using synthetic division, we have

-2	1	2	-9	-18	
		-2	0	18	
	1	0	-9	0	→ Remainder

Quotient = $x^2-9=(x+3)(x-3)$

Also one root is '–2', therefore one factor is $(x+2)$.

$\therefore\ x^3+2x^2-9x-18=(x+2)(x+3)(x-3)$

EXERCISE 2.4

1. Use synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.
   (i) $2x^3+7x^2-6x-8=0$; Factor: $x+4$
   (ii) $3x^3-5x^2+4x+2=0$; Factor: $3x+1$
   (iii) $x^3+2x^2-5x-6=0$ if 2 and -3 are its roots.
2. Find the value of k using synthetic division, if
   (i) 1 is the zero of the polynomial x^3-2kx^2+11.
   (ii) 3 is the zero of the polynomial $2x^3-kx^2+9$.
   (iii) When the polynomial $2x^3-kx^2+9$ is divided by $x-2$, the remainder is 14. Find the value of k.
3. Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are factors of the polynomial x^3+px^2+qx+6.
4. Find the values of a and b if 1 and 2 are the roots of the polynomial x^3-ax^2+bx-6.
5. Solve by using synthetic division.
   (i) 2 is root of the equation $x^3+x^2-7x+2=0$.
   (ii) 1 is root of the equation $x^2+4x-5=0$.

2.4 Simultaneous Equations

A system of equations having a common solution is called a system of simultaneous equations. The set of all the ordered pairs (x, y) which satisfies the system of equations is called the solution of the system. System of two equation involving two variables.

2.4.1 System of two Equations Involving two Variables

Case I: When one Equation is Linear and One Quadratic

If one of the equation is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following example.

Example 18: Solve the system of equations $x+y=7$ and $x^2-xy+y^2=13$

Solution: Given Equations are

$$x+y=7 \text{......}(i) \qquad x^2-xy+y^2=13 \text{............}(ii)$$

From equation (i)

$$x=7-y$$

Substituting in equation (ii)

$$(7-y)^2-(7-y)y+y^2=13$$
$$49-14y+y^2-7y+y^2+y^2=13$$
$$3y^2-21y+36=0$$
$$y^2-7y+12=0$$
$$y^2-4y-3y+12=0$$
$$y(y-4)-3(y-4)=0$$
$$(y-4)(y-3)=0$$
$$y-4=0 \text{ or } y-3=0$$
$$y=4, \; y=3$$

Putting $y=3$, in (i), we get $x=7-3=4$

Putting $y=4$, in (i), we get $x=7-4=3$

Hence solution set is $\{(4, 3), (3, 4)\}$

Case II: When both of the Equations are Quadratic

The equations in this case are classified as:

(i) Both the equations contain only x^2 and y^2 terms.

(ii) One of the equation is homogeneous in x and y.

(iii) Both the equations are non-homogeneous.

The methods of solving these types of equations are explained through the following examples.

Example 19: Solve the system of equations $x^2+y^2=25$ and $2x^2+3y^2=66$

Solution: Given equations are

$$x^2+y^2=25 \quad (i) \qquad 2x^2+3y^2=66 \quad (ii)$$

Multiplying equation (i) by 2 and subtracting from equation (ii).

$$\begin{array}{r} 2x^2+3y^2=66 \\ \pm 2x^2 \pm 2y^2=50 \\ \hline y^2=16 \end{array}$$

Putting $y^2=16$ in equation (i)

$x^2+16=25$

$x^2=9$

Now from $x^2=9$, we have $x=\pm 3$

and from $y^2=16$, we have $y=\pm 4$

Hence, solution set is $\{(\pm 3, \pm 4)\}$

Example 20: Solve the system of equations $x^2+y^2=20$ and $6x^2+xy-y^2=0$

Solution: Given equations are

$x^2+y^2=20 \qquad (i)$

$6x^2+xy-y^2=0 \qquad (ii)$

From equation (ii)

$-y^2+xy+6x^2=0$

$y^2-xy-6x^2=0$

$y^2-3xy+2xy-6x^2=0$

$y(y-3x)+2x(y-3x)=0$

$(y-3x)(y+2x)=0$

$y-3x=0$ or $y+2x=0$

$y=3x.......(iii)$ $\quad y=-2x.......(iv)$

Putting $y=3x$ in (i), $x^2+y^2=20$

$x^2+(3x)^2=20, \; x^2+9x^2=20$

$10x^2=20, \; x^2=2, \; x=\pm\sqrt{2}$

When $x=\sqrt{2}, \; y=3\sqrt{2}$

When $x=-\sqrt{2}, \; y=-3\sqrt{2}$

Putting $y=-2x$ in (i), $x^2+y^2=20$

$x^2+(-2x)^2=20, \; x^2+4x^2=20$

$5x^2=20, \; x^2=4, \; x=\pm 2$

When $x=2, \; y=-4$

When $x=-2, \; y=4$

Hence, solution set is $\{(\sqrt{2}, \; 3\sqrt{2}), (-\sqrt{2}, \; -3\sqrt{2}), (2,-4), (-2, \; 4)\}$

Test Taking Tip:

Most standardized tests have a time limit, so you must budget your time carefully. Some questions will be much easier than others. If you can not answer a question with in a few minutes go on to the next one. If there is still time left when you get to the end of the test, go back to the ones that you skipped.

EXERCISE 2.5

Solve the following simultaneous equations:

1. $2x+y=1; \quad x^2+y^2=10$
2. $3x-2y=1; \quad x^2+xy-y^2=1$
3. $3x+y+3=0; \quad (x+1)^2-4(x+1)-6=y$
4. $(x+3)^2-(y-2)^2=10; \quad x+y=4$
5. $x^2+3y^2=14; \quad 3x^2+y^2=6$
6. $2x^2-5y^2=8; \quad x^2+2y^2=13$
7. $2x^2-8xy+6y^2=0; \quad x^2+y^2=45$
8. $6x^2-5xy-y^2=0; \quad y^2+4xy=30$
9. $x^2+y^2+6x=1; \quad x^2+y^2+2(x+y)=3$
10. $x^2+y^2+6x+3y=3; \quad x^2+y^2+3x=3$

2.5 Real Problems Involving Quadratic Equations

There are many problems which lead to quadratic equations. To form an equation, we consider symbols for unknown quantities in the problems.

In order to solve problems, we must have

(i) Suppose the unknown quantities to be x or y.

(ii) Translate the problem into symbols and form the equation satisfying the conditions.

The method of solving the problem will be illustrated through the following examples.

Example 21: The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and breadth of the room.

Solution: Let the breadth of room = x metres

and the length of room = $x + 3$ metres

Area of the room = $x\,(x + 3)$ square metres

$x(x+3)=180$ By the conditions of the equation

$x^2+3x-180=0$

$(x+15)(x-12)=0$

$x=-15,\ x=12$

As breadth can not be negative, we take $x=12$.

Length $= x+3=12+3=15$

$\therefore$ Breadth of the room $=12m$ and length of room $=15m$

Example 22:

The sum of the Cartesian coordinates of a point is 6 and the sum of their squares is 20. Find the coordinate of the point.

Solution:

Let (x, y) be the coordinates of required point, then by the given condition we get

$x+y=6 \quad (i)$

$x^2+y^2=20 \quad (ii)$

From(i) $x+y=6 \quad \Rightarrow y=6-x$ putting in (ii)

$x^2+y^2=20 \quad \Rightarrow x^2+(6-x)^2=20$

$x^2+36-12x+x^2=20$

$x^2-6x+8=0$

$x^2-4x-2x+8=0$

$x(x-4)-2(x-4)=0$

$(x-4)(x-2)=0$

$x-4=0, \quad x-2=0$

$x=4, \quad x=2$

When $x=4,\ y=6-x=6-4=2$

When $x=2,\ y=6-x=6-2=4$

$\therefore$ The coordinates of the point are (4, 2) and (2, 4).

Problem Solving Plan:	
• Explore the problem.	• Plan the solution.
• Solve the problem.	• Examine the solution.

EXERCISE 2.6

1. Product of two consecutive even numbers is 120. Find the numbers.
2. The difference of a positive number and its square is 380. Find the number.
3. The difference of cubes of two numbers is 91. Find them.
4. The sum of the Cartesian coordinates of a point is 9 and the sum of their squares is 45. Find the coordinates of the point.
5. The sum of two numbers is 11 and the product is 30. Find the numbers.
6. The sum of the squares of two consecutive odd integers is 34. Find the integers.
7. The sum of ages of a father and his son is 50 years. Ten years ago, the father was 9 times as old as his son. Find the present age of father and son.
8. A two-digit number is decreased by 45 when the digits are reversed. If the sum of the digits is 11, find the number.
9. Sum of two numbers is 20. Find the numbers if the sum of first number and square of other is 40.

10. The reciprocal of the sum of reciprocals of two numbers is $\frac{12}{5}$.
Find the numbers if their sum is 10.

11. A group of 1025 students form two square patterns during morning assembly. One square pattern contains 5 more students than the other. Find the number of students in each pattern.

12. Sum of squares of two consecutive numbers is 145. The difference of their squares is 17. Find the numbers.

KEY POINTS

- In the quadratic formula, the expression under the radical sign '$b^2 - 4ac$', is called the discriminant. The discriminant tells the nature of roots of quadratic equation.
- The three roots of unity are: 1, ω, ω^2
- Sum of all the three cube roots of unity is zero. i.e. $1+\omega+\omega^2=0$
- The product of all the three cube roots of unity is 1.
  i.e. $1.\omega.\omega^2 = \omega^3 = 1$
- For any $n \in Z$, ω^n is equivalent to one of the cube roots of unity.
- Sum of the roots of a quadratic equation = $S = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coeficient of } x^2}$
- Product of roots of a quadratic equation = $P = \frac{c}{a} = \frac{\text{contant term}}{\text{coefficient of } x^2}$
- Formula used to form a quadratic equation when its roots are given is $x^2 - Sx + P = 0$
- Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.
- A system of equations having a common solution is called a system of simultaneous equations.
- The set of all the ordered pairs (x, y) which satisfies the system of equations is called the solution of the system.

REVIEW EXERCISE 2

1. **Encircle the correct option.**

(i) The discriminant of $ax^2+bx+c=0$ is

(a) b^2+4ac (b) b^2-4ac (c) $4ac-b^2$ (d) $-b^2-4ac$

(ii) If If α, β are the roots of $ax^2+bx+c=0$, then sum of roots is

(a) $\frac{c}{a}$ (b) $\frac{a}{c}$ (c) $-\frac{b}{a}$ (d) $\frac{a}{b}$

(iii) Roots of the equations $x^2 - 5x + 5 = 0$ are

(a) imaginary (b) rational (c) equal (d) irrational

(iv) If α, β are the roots of $7x^2 + 6x - 13 = 0$, the product of roots is

(a) $\frac{13}{7}$ (b) $-\frac{13}{7}$ (c) $\frac{6}{7}$ (d) $-\frac{6}{7}$

(v) Cube roots of 1 are

(a) $-1, \omega, \omega^2$ (b) $1, -\omega, \omega^2$ (c) $1, \omega, \omega^2$ (d) $1, \omega, \omega^3$

(vi) Sum of cube roots of unity is

(a) 0 (b) 1 (c) − 1 (d) 2

(vii) If ω is cube root of unity, then ω^{-12} is equal to

(a) ω (b) ω^2 (c) 1 (d) zero

(viii) Sum and product of roots of a quadratic equation are respectively 2 and 5. The equation is:

(a) $x^2 - 2x + 5 = 0$ (b) $x^2 + 2x + 5 = 0$

(c) $x^2 - 2x - 5 = 0$ (d) $x^2 + 2x - 5 = 0$

(ix) A number when added with its square gives 90. The number is:

(a) 6 (b) 9 (c) 10 (d) 12

(x) Value of $\left(1 + \omega + \omega^2\right)^{10}$ is

(a) zero (b) ω^3 (c) ω^2 (d) ω^4

2. For what values of m the roots of the equation are equal?
$(m-1)x^2 + 2mx + m + 3 = 0$. Also solve the equation.

3. If $x - 1$ and $x + 2$ are factors of $x^4 - 5x^2 + 4$, using synthetic division, find the other two factors.

4. If α, β are the roots of $ax^2 + bx + c = 0$, find the value of $(\alpha - 3)(\beta - 3)$.

5. Solve the simultaneous equations: $x^2 - y^2 = 5$, $4x^2 - 3xy = 11$

6. If roots of $25x^2 - 5ax - b = 0$ are equal then find the values of a and b if $a^2 + b = 6$.

7. A rectangular chocolate box has volume as $x^3 + 2x^2 - 5x - 6$. Find the length and width of the box if its height is $x - 2$.

UNIT 03 Variations

In this unit the students will be able to:

- **Know about ratio, proportion (variation) and types of proportion (variation).**
- **Find the 3rd, 4th, mean and continued proportion.**
- **Apply theorems on proportion to find proportion.**
- **Know joint variation and solve real-life problems related to joint variation.**
- **Use K- Method to prove conditional equations involving proportion.**
- **Solve real life problems based on variation.**

The LEM (Lunar Exploration Module) used by astronauts to explore the moon's surface during the Apollo space missions weights about 30,000 pounds on Earth. On the moon, there is less gravity so it weight less, meaning that less fuel is needed to lift off from the moon's surface. The foure of gravity on Earth is about six times as much as that on the moon. The relationship of two gravities can be expressed in the equation $y = 6x$, where y represents the weight on Earth and x represents the weight on the moon. The 6 is called the constant of variation. Find how much LEM weights on the moon. The relationships described above is an example of direct variation.

3.1 Ratio, Proportion (Variation) and Types of Proportion

3.1.1 Ratio

Ratio is a mathematical term used for comparing two (or more) quantities having the same units. It indicates how many times one quantity is of the other.

> Ratio is the comparison of two quantities say a and b, written as $a : b$ where $b \neq 0$.
> We read $a : b$ as 'a ratio b' or a is to b or a to b.

Key Facts

- Ratio is expressed in its simplest form with no unit. First element of the ratio is called antecedent and its second element is called consequent.
- In $a : b$, a is ***antecedent*** and b is ***consequent.***
- In general $a : b \neq b : a$ and $a : b = c : d$ if and only if $a = c$ and $b = d$.
- Just like fractions, ratio is expressed in lowest form and its elements are integers with no common factor other than number 1.

3.1.2 Proportion

A statement which shows the equality of two ratios is called a proportion.

If $\frac{a}{b}$ and $\frac{c}{d}$ are two ratios such that $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c}{d}$ is called a proportion. This proportion is read in two different ways:

I. a is to b as c is to d.

II. 'a to b' is proportional to 'c to d'.

Each of a, b, c and d are called the terms of the proportion.

The first and fourth terms of the proportion (i.e. a and d) are called extremes and the second and third terms (i.e. b and c) are called means.

Now consider $\frac{a}{b} = \frac{c}{d}$

$$\frac{a}{b} \times bd = \frac{c}{d} \times bd$$

or $\quad ad = cb$

i.e. the product of extremes = the product of means.

By using the above relationship between the product of means and extremes, we can find the missing term of the proportion when any three terms of proportion are given.

Example 1: Find the value of y in the proportion $\frac{2}{y+3} = \frac{7}{8}$.

Solution: $\frac{2}{y+3} = \frac{7}{8}$

$$2 \times 8 = 7(y+3)$$
$$16 = 7y + 21$$
$$16 - 21 = 7y$$
$$-5 = y$$

Hence, required value of y is -5.

Key Facts

> Proportions are helpful in solving many problems.
> In recipes these are used when a larger batch of ingredients is used than the recipe calls for. It is also used when cement and concrete are mixed for construction purposes. We draw maps also by taking help from the concept of proportion, when we have to decrease the length on map as compared to original.

EXERCISE 3.1

1. What should be added to each term of a ratio 5 : 9, so that it becomes equivalent to 10 : 14?
2. If $a+b:a-b=7:1$, then find $\frac{1}{a}:\frac{1}{b}$.
3. Two numbers differ by 25 and ratio between the numbers is 3 : 5. Find the numbers.
4. If 8 is added to each number of a ratio 5 : 7, we get a new ratio 9 : 11. Find the numbers.
5. Set up all possible proportions using the numbers: 5, 6, 10 and 12.
6. If $\frac{3y-1}{7}:\frac{3}{5}::\frac{2y}{3}:\frac{7}{5}$. Find y.
7. If $\frac{a-3}{2}:\frac{5}{a-1}::\frac{a-1}{3}:\frac{4}{a+4}$. Find a.
8. Find a, if $(x-y):x^2-y^2::a:(x+y)$

3.1.3 Types of Proportions (Variations)

Proportions (variations) are of two types.

I. Direct proportion or variation **II.** Inverse proportion or variation

I. Direct Proportion

It is a relation between two quantities such that when one quantity is increased (decreased), the other quantity is also increased (decreased) in the same ratio.

For example, if one quantity is doubled, the other quantity is also doubled.

Explanation: Many people go to some cold places to spend their summer vacations. If you plan to go to Murree to spend your vacations and want to stay in a hotel. You may take help from the following table for estimating your charges according to the number of days, you stay there.

No of days (x)	1	2	3	4	5	6	7	8
Charges (y) (in Rs.)	500	1000	1500	2000	2500	3000	3500	4000

We can observe that increase in number of days causes increase in the charges and vice versa. Hence, we can say that the number of days is directly proportional to the charges.

If we denote number of days by a variable x and charges by another variable y, then we say that y is directly proportional to x.

Symbolically, we write $y \propto x$. The sign '$\propto$' is called '*sign of proportionality*' or '*sign of variation*' and is read as '*varies as*'.

In the above example

$$\frac{y}{x}=\frac{500}{1}=\frac{1000}{2}=\frac{1500}{3}=\frac{2000}{4}=\ldots\ldots=500=\text{constant}$$

Therefore, when $y \propto x$, the ratio $\frac{y}{x}$ is constant and is known as '*constant of proportionality*' or '*constant of variation*'. If we denote this constant by k, then

$$\frac{y}{x}=k \quad \text{or} \quad y=kx \ldots\ldots\ldots\ldots (1)$$

In the above example, $k = 500$ and thus the above equation (1) becomes

$$y = 500\,x \ldots\ldots\ldots\ldots\ldots (2)$$

Equation (2) is a relation between x and y where y varies directly as x. Using this if one of the quantities x or y is given, the other quantity can be found.

Direct variations are applied for solving many real life problems.

Some examples of direct variations are:

(i) Number of men and quantity of food
(ii) Number of items sold and their price
(iii) Radius and area of circle
(iv) Time and distance
(v) Work and time etc.

Example 2: If $y \propto x$ and y = 12 when x = 8. Find the value of y when x = 24.

Solution: As $y \propto x$. Therefore

$$\frac{y}{x}=k \quad \text{or} \quad y=kx \ldots\ldots\ldots\ldots\text{(i)}$$

Putting $x = 8$ and y = 12 in equation (i).

$$12 = k \times 8$$

or $$k=\frac{12}{8}=\frac{3}{2}$$

Putting $x = 24$ and $k=\frac{3}{2}$ in equation (i).

$$y=\frac{3}{2}\times 24=36$$

Example 3: If y varies directly as $x + 3$.

(i) Find an expression for y in term of x.

(ii) It is given that y = 15 when x = 2. Find x when 2y = 10.

Solution: (i) As y varies directly as $x + 3$. Therefore

$$\frac{y}{x+3}=k \quad \text{or} \quad y=k(x+3) \ldots\ldots\ldots\ldots\text{(i)}$$

(ii) Putting $x = 2$ and y = 15 in equation (i).

$$15 = k\,(2 + 3)$$

or $\quad k = \frac{15}{5} = 3$

Now $2y = 10$ implies that $y = 5$.

$\therefore$ Putting $k = 3$ and $y = 5$ in equation (i)

$$5 = 3\,(x + 3)$$

$$5 = 3x + 9$$

or $\quad 3x = 5 - 9$

$$x = -\frac{4}{3}$$

Example 4: Given that $a = 27$ and $b = 18$. If b is directly proportional to the cube root of a, then find b when $a = 64$.

Solution: As $b \propto \sqrt[3]{a}$

$\therefore b = k\sqrt[3]{a}$ …………..(i)

Putting $a = 27$ and $b = 18$ in equation (i).

$$18 = k\sqrt[3]{27}$$

$$18 = k \times 3$$

or $\quad k = \frac{18}{3} = 6$

Now putting $a = 64$ and $k = 6$ in equation (i).

$$b = 6\sqrt[3]{64} = 6 \times 4 = 24$$

Example 5: A train travels 240 km in 3 hours. In how much time will the train cover 560 km? What does constant k represent here?

Solution: If distance is represented by s and time by t, then:

$$s \propto t.$$

$\therefore \quad s = kt$ ……….(i)

Putting $s = 240km$ and $t = 3h$ in equation (i).

$$240 = k \times 3$$

or $\quad k = \frac{240}{3} = 80$

Putting $s = 560km$ and $k = 80$ in equation (i).

$$560 = 80 \times t$$

$$t = \frac{560}{80} = 7$$

Therefore, train will travel 560 *km* in 7 hours.

The constant k represents the speed of train which is 80 *km /h*.

Check Point

- A map is scaled so that 1cm represents 15km. How for a part are two towns if they are 7.9cm apart on the map?
- 6 feet of steel wire weighs 0.7kg. How much does 100 feet of the same steel wire weigh?

EXERCISE 3.2

1. If $y \propto x$ and $y = 36$ when $x = 4$. Find the value of the constant of variation.
2. If $y \propto x$ and $y = 15$ when $x = 3$. Find the value of y when $x = 20$.
3. If $y \propto x$ and $y = 10$ when $x = 4$. Find the value of x when $y = 14$.
4. If $y \propto x^2$ and $y = 28$ when $x = 2$. Find the value of y when $x = 7, 10$.
5. If $y \propto x^3$ and $y = 125$ when $x = 5$. Find the value of x when $y = 27$.
6. If $y + 1$ varies directly as $2x$.
   (i) Find an expression for y in term of x.
   (ii) It is given that $y = 12$ when $x = 3$. Find y when $x = 8$.
7. Given that $a = 36$ and $b = 9$. If b is directly proportional to the square root of a, then find
   (i) b when $a + 1 = 50$. (ii) a when $b = 81$.
8. If $y^2 \propto x$ and $y = 25$ when $x = 5$. Find the value of y when $x = 9$.
9. In a river, speed (v) of water on the surface is directly proportional to the square of height (h) from bottom. If $v = 100$ *m/s* for $h = 10$ *m*. Find the speed of water at the height of 12 *m*.
10. If 15 people produce 50 items daily. How many people are needed to produce 250 items daily at the same rate?
11. Price of 5 dozen eggs is Rs. 450. What is the price of 50 eggs?
12. If $y \propto x^2$, then complete the following table.

x	2	4	5			3	0
y	12			108	300		

II. Inverse Proportion

It is a proportion in which increase/decrease in one quantity causes decrease/increase in the other in the same ratio and vice versa.

For example, if one quantity is doubled, the other quantity is halved in the same ratio.

Explanation:
The time taken by a jeep to travel a distance of 120 km at different speeds is calculated in the following table.

Speed in km/h (y)	120	60	40	30	20
Time in hours (x)	1	2	3	4	6

We can observe that decrease in speed causes increase in the number of hours and vice versa. Hence, we can say that the speed is inversely proportional to the time.
If we denote speed by a variable y and hours by variable x, then y is inversely proportional to x.

Symbolically: $y \propto \frac{1}{x}$.

In the above table

$$x \times y = 1 \times 120 = 2 \times 60 = 3 \times 40 = 4 \times 30 = 20 \times 6 = 120 = \text{constant}$$

Therefore, when $y \propto \frac{1}{x}$, the product xy is constant.

If we denote this constant by k, then

$$xy = k \text{ .... } (3)$$

In the above example, $k = 120$ and thus equation (3) becomes

$$xy = 120 \text{................}(4)$$

Equation (3) is relation between x and y when y varies inversely as x. Using this if one of the quantities x or y is given, the other quantity can be found.
Inverse variations are applied for solving many real life problems.
Some examples of inverse variation are:

- **(i)** Number of people and days for work
- **(ii)** Speed of bike and time
- **(iii)** Number of people and quantity of food for fixed days
- **(iv)** Pressure and volume

Example 6: If $y \propto \frac{1}{x}$ and y = 6 when $x = 4$. Find the value of y when $x = 12$.

Solution: As $y \propto \frac{1}{x}$.

$\therefore xy = k$ or $k = xy$ ....(i)

Putting $x = 4$ and y = 6 in equation (i).

$k = 4 \times 6 = 24$

Putting $x = 12$ and $k = 24$ in equation (i).

$24 = 12 \times y$

$y = \frac{24}{12} = 2$

Example 7: If y varies inversely as $x + 1$.

(i) Find an expression for y in terms of x.

(ii) It is given that y = 15 when $x = 2$. Find x when $y - 1 = 8$.

Solution: **(i)** As y varies inversely as $x + 1$.

$\therefore k = y(x+1)$ ....(i)

(ii) Putting $x = 2$ and y = 15 in equation (i).

$k = 15(2+1) = 45$

Now $y - 1 = 8$ implies that $y = 9$.

$\therefore$ Putting $k = 45$ and $y = 9$ in equation (i).

$45 = 9(x+1)$

$x + 1 = \frac{45}{9} = 5$

$x = 5 - 1 = 4$

Example 8: Given that $a = 25$ and $b = 4$. If b is inversely proportional to the square root of a, then find b when $a = 16$.

Solution: As $b \propto \frac{1}{\sqrt{a}}$

$\therefore b\sqrt{a} = k$ …………..(i)

Putting $a = 25$ and $b = 4$ in equation (i).

$4\sqrt{25} = k$

$k = 4 \times 5 = 20$

Putting $a = 16$ and $k = 20$ in equation (i).

$b\sqrt{16} = 20$

$b \times 4 = 20$

$b = \frac{20}{4} = 5$

Challenge

When air is pumped into a tyre, the pressure required varies inversely as the volume of the air. If the pressure is 30 lb^3/in^2, when the volume is 140 in^3, find the pressure when the volume is $100 in^3$.

Example 9: A car travels at a speed of 100 *km/h* to cover a distance in 3 hours. Find the speed of car to cover the same distance in 4 hours.

Solution: If speed is represented by v and time by t, then

$v \propto \frac{1}{t}$

$\therefore v\,t = k$ …………..(i)

Putting $v = 100$ *km/h* and $t = 4$ *h* in equation (i).

$100 \times 3 = k$

or $k = 300$

Putting $t = 4$ *h* and $k = 300$ in equation (i).

$v \times 4 = 300$

$v = \frac{300}{4} = 75$

Therefore, speed of train is 75 *km/h*.

EXERCISE 3.3

1. If $y \propto \frac{1}{x}$ and y = 10 when x = 4. Find the value of the constant of variation.

2. If $y \propto \frac{1}{x}$ and y = 6 when x = 3. Find the value of y when x = 20.

3. If $y \propto \frac{1}{x}$ and y = 12 when x = 5. Find the value of x when y = 12.

4. If $y \propto \frac{1}{x^2}$ and y = 7 when x = 2. Find the value of y when x = 7, 14.

5. If $y \propto \frac{1}{x^3}$ and y = 2 when x = 3. Find the value of x when y = 6.

6. If 2y varies inversely as $x + 1$.

(i) Find an expression for y in term of x.

(ii) It is given that y = 10 when x = 4. Find y when x = 8.

7. Given that $a = 27$ and $b = 3$. If b is inversely proportional to the cube root of a, then find

(i) b when $3a-1 = 14$. (ii) a when $2b = 8$.

8. If $y^2 \propto \frac{1}{x}$ and y = 5 when x = 10. Find the value of y when x = 12.

9. Speed of air plane is inversely proportional to the time. If v = 100 m/s for t = 15 seconds. Find the speed of air plane to reach the target in 20 seconds.

10. 15 workers can build a wall in 25 days.

(i) How many workers are needed to complete the wall in 10 days?

(ii) In how many days will 12 people complete that job?

11. The volume of a gas at a pressure of 100 Pa is 15 m^3. Find the volume of the gas at a pressure of 150 Pa.

12. If $y \propto \frac{1}{x}$, then complete the following table.

x	12	9	6			3	1
y	3			4	12		

3.2 3rd, 4th, Mean and Continued Proportion

3.2.1 Fourth Proportion

If quantities *a, b, c* and *d* are in proportion then *d* is called the *fourth proportional* to *a, b* and *c*.

Mean, Third and Continued Proportion

If we put $b = c$ in the proportion $a : b :: c : d$, then $a:b::b:d$.
Here *b* is the *mean proportional* to *a* and *d*, and *d* is the *third proportional* to *a* and *b*.
Moreover *a, b* and *d* are in *continued proportion*.

Example 10: Find the fourth proportional to

(i) 3, 4, 5 (ii) $a^2-b^2,\ a-b,\ a+b$

Solution: (i) Let "*x*" be the fourth proportional then

$$3:4::5:x \text{ or } \frac{3}{4}=\frac{5}{x}$$
$$3x=20$$
$$x=\frac{20}{3}$$

So, the fourth proportional is $\frac{20}{3}$.

(ii) Let "*x*" be the fourth proportional then

$$a^2-b^2:a-b::a+b:x$$
$$\frac{a^2-b^2}{a-b}=\frac{a+b}{x}$$
$$x=\frac{(a+b)(a-b)}{a^2-b^2}$$
$$x=\frac{a^2-b^2}{a^2-b^2} \qquad \therefore (a-b)(a+b)=a^2-b^2$$
$$x=1$$

So, the fourth proportional is 1.

> **Challenge**
> - Is it possible to wirte a proportion using the numbers 3, 4, 6, and 8? Explain your reasoning.
> - If $\frac{a}{b}=\frac{c}{d}$ for nonzero numbers a, b, c and d, is it also true that $\frac{a}{c}=\frac{b}{d}$? Explain.

Example 11: Find the third proportional to

(i) 3, 8 (ii) $(a-b)^3, a-b$

Solution:

(i) Let "*x*" be the third proportional then

$$3:8::8:x$$
$$\frac{3}{8}=\frac{8}{x}$$
$$3x=64$$
$$x=\frac{64}{3}$$

So, the third proportional is $\frac{64}{3}$.

(ii) Let "x" be the third proportional then

$$(a-b)^3 : a-b :: a-b : x$$

$$\frac{(a-b)^3}{a-b} = \frac{a-b}{x}$$

$$x(a-b)^3 = (a-b)^2$$

$$x = \frac{(a-b)^2}{(a-b)^3} = \frac{1}{a-b}$$

So, the third proportional is $\frac{1}{a-b}$.

Example 12: Find the mean proportional of

(i) 5, 20 **(ii)** $\frac{x^2+5x+4}{x-2}, \frac{x^2+2x-8}{x+1}$

Solution: (i) If "y" be the mean proportional then we can write

$$3 : y :: y : 8$$

$$\frac{3}{y} = \frac{y}{8}$$

$$y^2 = 24$$

$$y = \pm\sqrt{24}$$

So, the mean proportional is $\sqrt{24}$.

(ii) If "y" be the mean proportional then we can write

$$\frac{x^2+5x+4}{x-2} : y :: y : \frac{x^2+2x-8}{x+1}$$

Product of means = Product of extremes

$$y^2 = \frac{x^2+5x+4}{x-2} \times \frac{x^2+2x-8}{x+1}$$

$$y^2 = \frac{(x+4)(x+1)}{x-2} \times \frac{(x-2)(x+4)}{x+1}$$

$$y^2 = (x+4)^2$$

$$y = \pm(x+4)$$

So the mean proportional = $\pm(x+4)$

EXERCISE 3.4

1. Find the fourth proportional.

(i) $5,\ 10,\ 20$ (ii) $8,\ 16,\ 32$ (iii) $6y^3, 3y^2,\ 8y^3$

(iv) $10x^4y^5, 5xy^4,\ 20x^2y^2$ (v) $y^2-11y+24,\ y-3,\ y^3-y^2$

(vi) $a^3+b^3,\ a^2-b^2,\ a^2-ab+b^2$

2. Find the third proportional.

(i) $12,\ 24$ (ii) $x^4, 4x^3$ (iii) $(a-b)^2,\quad a^3-b^3$

(iv) $\dfrac{x^2-y^2}{x^2+y^2},\ \dfrac{x-y}{x^2-xy+y^2}$

3. Find mean proportional between

(i) $40,\ 90$ (ii) $30x^4y^2z^2, 270y^4z^6$ (iii) $a^2-b^2, \dfrac{a-b}{a+b}$

(iv) $\dfrac{x^3+y^3}{x^2-y^2},\quad \dfrac{x-y}{x^2-xy+y^2}$

4. Find the value of y in the following continued proportion.

(i) $10,\ y,\ 90$ (ii) $4,\ y,\ 9$ (iii) $8,\ y-4,\ 32$

(iv) $9, 3y-6,\ 36$

3.3 Theorems of Proportion

If four quantities *a, b, c* and *d* from a proportion then $a : b = c : d$.
We know that in any proportion:

product of extremes = product of means

$$\therefore\quad ad = bc \ \ldots\ldots\ldots\ldots \text{(i)}$$

(i) Theorem of Invertendo

If $a : b = c : d$ then $b : a = d : c$

Proof: If we divide both sides of equation (i) by "*ac*" then:

$$\frac{bc}{ac}=\frac{ad}{ac}$$
$$\Rightarrow \frac{b}{a}=\frac{d}{c}$$
$$\Rightarrow b:a=d:c$$

Example 13: If $5x:7y=35:4y$ then $7y:5x::4y:35$

Solution: We have $5x:7y=35:4y$ or $\dfrac{5x}{7y}=\dfrac{35}{4y}$

By Invertendo Property

$$\frac{7y}{5x}=\frac{4y}{35}$$

$$7y:5x::4y:35$$

(ii) Theorem of Alternendo

If $a:b=c:d$ then $a:c=b:d$

Proof: If we divide both sides of equation (i) by "*cd*" then:

$$\frac{ad}{cd}=\frac{bc}{cd}$$

$$\Rightarrow \frac{a}{c}=\frac{b}{d}$$

$$\Rightarrow a:c=b:d$$

Example 14: If $x:y+2=25:3t+4$ then $x:25=y+2:3t+4$

Solution: We have $x:y+2=25:3t+4$

By Alternando Property

$x:25=y+2:3t+4$

(iii) Theorem of Componendo

If $a:b=c:d$ then $a+b:b=c+d:d$

Proof: Adding 1 to both sides of Eq. $\frac{a}{b}=\frac{c}{d}$

$$\frac{a}{b}+1=\frac{c}{d}+1$$

Taking L.C.M on both sides

$$\frac{a+b}{b}=\frac{c+d}{d}\ldots\ldots\ldots\ldots(ii)$$

$$\Rightarrow a+b:b=c+d:d$$

Example 15: If $x+1:y+2=s-1:t-2$ then prove that $x+y+3:y+2=s+t-3:t-2$

Solution: We have,

$x+1:y+2=s-1:t-2$

By Componendo Theorem

$(x+1)+(y+2):(y+2)=s-1+t-2:t-2$

$x+y+3:y+2=s+t-3:t-2$

(iv) Theorem of Dividendo

If $a:b=c:d$ then $a-b:b=c-d:d$

Proof: Subtracting 1 from both sides of Eq. $\frac{a}{b}=\frac{c}{d}$

$$\frac{a}{b}-1=\frac{c}{d}-1$$

Taking L.C.M

$$\frac{a-b}{b}=\frac{c-d}{d}\ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

$$a-b:b=c-d:d$$

Check Point

Prove theorem of dividendo by taking example 15.

(v) Theorem of Componendo-Dividendo

If $a:b=c:d$ then $a+b:a-b=c+d:c-d$

Proof: Dividing equation (ii) by equation (iii), we get:

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}}=\frac{\frac{c+d}{d}}{\frac{c-d}{d}}$$

$$\Rightarrow \frac{a+b}{a-b}=\frac{c+d}{c-d}$$

$$\Rightarrow a+b:a-b=c+d:c-d$$

This property is called Componendo-Dividendo and it is among the most commonly known technique in mathematics for solving many problems.

Example 16: If $a:b=c:d$ then prove that

$$5a+7b:5a-7b::5c+7d:5c-7d$$

Solution: Given that $a:b=c:d$ or $\frac{a}{b}=\frac{c}{d}$

Multiplying both the sides by $\frac{5}{7}$ we get,

$$\frac{5a}{7b}=\frac{5c}{7d}$$

Using Componendo-Dividendo Theorem

$$\frac{5a+7b}{5a-7b}=\frac{5c+7d}{5c-7d}$$

$$\therefore\ 5a+7b:5a-7b::5c+7d:5c-7d$$

Example 17: If $y=\frac{6ab}{a+b}$ then find the value of $\frac{y+3a}{y-3a}+\frac{y+3b}{y-3b}$

Solution: Given that $y=\frac{6ab}{a+b}$ ....(i)

$$y=\frac{3a\times 2b}{a+b}$$

$$\therefore\ \frac{y}{3a}=\frac{2b}{a+b}$$

Using Componendo-Dividendo Theorem

$$\frac{y+3a}{y-3a}=\frac{2b+a+b}{2b-a-b}$$

$$\frac{y+3a}{y-3a}=\frac{a+3b}{-a+b} \text{...............}(ii)$$

Again from Eq. (i)

$$y=\frac{2a\times 3b}{a+b}$$

$$\therefore\ \frac{y}{3b}=\frac{2a}{a+b}$$

Using Componendo-Dividendo Theorem

$$\frac{y+3b}{y-3b}=\frac{2a+a+b}{2a-a-b}$$

$$\frac{y+3b}{y-3b}=\frac{3a+b}{a-b} \text{ ....}(iii)$$

Adding equations (*ii*) and (*iii*)

$$\frac{y+3a}{y-3a}+\frac{y+3b}{y-3b}=\frac{a+3b}{-a+b}+\frac{3a+b}{a-b}$$

$$=\frac{-a-3b}{a-b}+\frac{3a+b}{a-b}$$

$$=\frac{-a-3b+3a+b}{a-b}$$

$$=\frac{2a-2b}{a-b}=\frac{2(a-b)}{a-b}=2$$

$$\therefore \frac{y+3a}{y-3a}+\frac{y+3b}{y-3b}=2$$

Example 18:

Solve the following equation by using Componendo-Dividendo property.

$$\frac{(x+7)^2+(x-5)^2}{(x+7)^2-(x-5)^2}=\frac{5}{4}$$

Solution: By using Componendo-Dividendo Theorem, we have

$$\frac{(x+7)^2+(x-5)^2+(x+7)^2-(x-5)^2}{(x+7)^2+(x-5)^2-(x+7)^2+(x-5)^2}=\frac{5+4}{5-4}$$

or $$\frac{2(x+7)^2}{2(x-5)^2}=\frac{9}{1}$$

or $$\left(\frac{x+7}{x-5}\right)^2=9$$

or $$\frac{x+7}{x-5}=\pm 3$$

$\Rightarrow \quad \frac{x+7}{x-5}=3$	or $\quad \frac{x+7}{x-5}=-3$
$\Rightarrow \quad x+7=3(x-5)$	$x+7=-3(x-5)$
$\Rightarrow \quad x+7=3x-15$	$x+7=-3x+15$
$\Rightarrow \quad 7+15=3x-x$	$x+3x=15-7$
$\Rightarrow \quad 22=2x$	$4x=8$
$\Rightarrow \quad x=11$	$x=2$

$\therefore$ **Solution set** $=\{2, 11\}$

EXERCISE 3.5

1. Using the theorems of componendo-dividendo, prove that $a : b = c : d$, if

(i) $\dfrac{3a+5b}{3a-5b}=\dfrac{3c+5d}{3c-5d}$ (ii) $\dfrac{a^3+3ab^2}{3a^2b+b^3}=\dfrac{c^3+3cd^2}{3c^2d+d^3}$

(iii) $\dfrac{5a+7b}{5a-7b}=\dfrac{5c+7d}{5c-7d}$ (iv) $(9a+10b)(9c-10d)=(9a-10b)(9c+10d)$

2. Using the theorem of componendo-dividendo, find the value of:

(i) $\dfrac{a+5b}{a-5b}+\dfrac{a+5c}{a-5c}$, if $a=\dfrac{10bc}{b+c}$ (ii) $\dfrac{y+2\sqrt{7}}{y-2\sqrt{7}}-\dfrac{y+2\sqrt{2}}{y-2\sqrt{2}}$, if $y=\dfrac{4\sqrt{14}}{\sqrt{7}+\sqrt{2}}$

(iii) $\dfrac{a+3x}{a-3x}+\dfrac{a+2y}{a-2y}$, if $a=\dfrac{6xy}{x+y}$

3. Solve the following:

(i) $\dfrac{(x-3)^2-(x-5)^2}{(x-3)^2+(x-5)^2}=\dfrac{4}{5}$ (ii) $\dfrac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}}=2$

(iii) $\dfrac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3}=\dfrac{13}{14}$

4. If $p=\dfrac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$, then prove that $p^2y-2px+y=0$.

3.4 Joint Variation

If a variable varies directly as the product of several others, we say that it varies jointly as the others.
Thus if $y = kxwz$ we say that "*y*" varies jointly as *x*, *w* and z.
If a variable "*y*" varies directly as "*x*" and varies inversely as "*z*"

Then, $y \propto x$ and $y \propto \dfrac{1}{z}$

In joint variation, we write it as

$$y \propto \frac{x}{z} \Rightarrow y = k\frac{x}{z}$$

where, $k \neq 0$ is the constant of variation.

Example 19:

Determine the constant of variation if L varies jointly as a & m and inversely as c and is 27 for $a=3$, $m=6$ and $c=2$.

Solution: Given that: $L \propto a$, $L \propto m$ and $L \propto \dfrac{1}{c}$.

In joint variation, it can be written as

Check Point

The volume of any gas varies inversely with its pressure as long as the temprutare remains constan. If a helium filled ballon has a volume of 3.4 cubics decimeters at a pressure of 120 kilopascals, what is its volume at 101.3 kilomascles?

$$L \propto \frac{am}{c} \quad \text{or} \quad L = k\frac{am}{c} \text{...........}(i)$$

Putting the values of *L, a, m* and *c* in Eq. (i), we get

$$27 = k\frac{(3)(6)}{2}$$

$$27 = k(9)$$

$$k = \frac{27}{9} = 3$$

Example 20:

The safe load of a rectangular beam of given width varies directly as the square of the depth of the beam and inversely as the length between supporters. If the safe load for a beam of given width that 10 feet long and 5 inches deep is 2400 Pounds, determine the safe load for a beam of the same material and width if it is 8 inches deep and 16 feet between supporters.

Solution: We shall assume that

d = depth of beam in inches
s = length between supporters in feet
l = safe load in pounds

Then, $l = \frac{kd^2}{s}$ ....(i)

Putting $s = 10$ feet, $d = 5$ inches and $l = 2400$ pounds in equation (i), we obtain

$$2400 = \frac{k(5)^2}{10}$$

$$k = 960$$

$\therefore$ Equation (i) becomes

$$l = \frac{960d^2}{s}$$

Now, putting $d = 8$ inches and s = 16 feet, we have:

$$l = \frac{960(8)^2}{16}$$

$$l = 3840$$

Therefore, the safe load (l) is 3840 pounds.

3.5 K-Method

Let $a : b :: c : d$ be a proportion, then $\frac{a}{b} = \frac{c}{d}$.

Let

$$\frac{a}{b} = \frac{c}{d} = k$$

$$\Rightarrow \frac{a}{b} = k \quad \text{and} \quad \frac{c}{d} = k$$

$\Rightarrow a = bk$ and $c = dk$

These equations are helpful to solve problems relating to proportion more easily. This method is called K-Method. To understand the method, let's solve some examples.

Example 21:

If $\dfrac{a}{b} = \dfrac{c}{d}$ then prove that $\dfrac{2ac+3bd}{2ac-3bd} = \dfrac{2a^2+3b^2}{2a^2-3b^2}$

Solution:

Let $\dfrac{a}{b} = \dfrac{c}{d} = k$, then $a = bk, c = dk$

Taking L.H.S

$$\frac{2ac+3bd}{2ac-3bd} = \frac{2(bk)(dk)+3bd}{2(bk)(dk)-3bd}$$

$$= \frac{2bdk^2+3bd}{2bdk^2-3bd} = \frac{bd(2k^2+3)}{bd(2k^2-3)}$$

$$= \frac{2k^2+3}{2k^2-3}$$

Now taking R.H.S

$$\frac{2a^2+3b^2}{2a^2-3b^2} = \frac{2(bk)^2+3b^2}{2(bk)^2-3b^2}$$

$$= \frac{2b^2k^2+3b^2}{2b^2k^2-3b^2} = \frac{b^2(2k^2+3)}{b^2(2k^2-3)} = \frac{2k^2+3}{2k^2-3}$$

Hence, L.H.S = R.H.S

> **Check Point**
>
> Every 10 minutes, the cashier at the head of your line helps 3 people. There are 11 people in line of you what a proportion that can be used to determine how long you will have to wait to purchase ticket?

Example 22:

If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ then prove that $\dfrac{a^3+b^3+c^3}{x^3+y^3+z^3} = \dfrac{abc}{xyz}$

Solution: Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$, then $a = bk, c = dk$ and $e = fk$

Taking L.H.S

$$\frac{a^3+b^3+c^3}{x^3+y^3+z^3} = \frac{a^3+b^3+c^3}{(ak)^3+(bk)^3+(ck)^3}$$

$$= \frac{a^3+b^3+c^3}{a^3k^3+b^3k^3+c^3k^3}$$

$$= \frac{a^3+b^3+c^3}{k^3(a^3+b^3+c^3)} = \frac{1}{k^3}$$

Now taking R.H.S

$$\frac{abc}{xyz} = \frac{abc}{(ak)(bk)(ck)}$$

$$= \frac{abc}{k^3(abc)} = \frac{1}{k^3}$$

Hence, L.H.S = R.H.S

EXERCISE 3.6

1. If d varies jointly as a, b^2, c and $d = 5$, when $a = 2, b = 3, c = 10$. Find d when $a = 4, b = 7,$ and $c = 3$.
2. If V varies directly as a^2 and inversely as the product bc^2. If $V = 2$ when $a = 8$, b = 7 and c = 2. Find the value of V when $a = 6$, $b = 3$ and c = 1.
3. If u varies directly as the product of $x\,y^3$ and inversely as z^2 and $u = 27$ when $x = 7,\ y = 6,$ and $z = 7$. Find the value of u when $x = 6,\ y = 2,$ and $z = 3$.
4. The area A of a triangle varies jointly as the base b and height h. Find the equation of joint variation if $A = 100\text{cm}^2$, $b = 25$cm and $h = 8$cm.
5. The area of a trapezoid varies jointly as the height h and sum of the lengths of parallel sides S. If area of the trapezoid is 20m^2, its height is 5m and bases are 3m and 5m, find the constant of variation. Then write the general equation for the area of trapezoid.
6. When air is pumped into a fire, the required pressure varies inversely as the volume of the air. If the pressure is 30 lb/in^2 when the volume is 140 in^3, find the pressure when the volume is 100 in^3.
7. If $w : x :: y : z$, then show that:

   (i) $\dfrac{6w-5x}{6w+5x} = \dfrac{6y-5z}{6y+5z}$ (ii) $\dfrac{w}{x} = \sqrt[3]{\dfrac{5w^3-10y^3}{5x^3-10z^3}}$

   (iii) $w^2+x^2 : \dfrac{w^3}{w+x} = y^2+z^2 : \dfrac{y^3}{y+z}$
8. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, then:

   (i) $\dfrac{2x}{a} = \dfrac{\sqrt{4x^2+4y^2+4z^2}}{\sqrt{a^2+b^2+c^2}}$ (ii) $\dfrac{3x^3+5y^3+7z^3}{3a^3+5b^3+7c^3} = \left(\dfrac{3x+5y+7z}{3a+5b+7c}\right)^3$

KEY POINTS

- Ratio is the comparison of two quantities of the same kind.
- First element of the ratio is called antecedent and second element is called consequent.
- A statement which shows the equality of two ratios is called a proportion.
- **Direct proportion:** It is a proportion in which increase (decrease) in one quantity causes increase (decrease) in the other in the same ratio.
- **Inverse proportion:** It is a proportion in which increase (decrease) in one quantity causes decrease (increase) in the other in the same ratio.
- If quantities a, b, c and d are in proportion then, d is called the fourth proportional to a, b and c.
- If we put $b = c$ in the proportion $a:b::c:d$, then $a:b::b:d$. Here b is the mean proportional to a and d, and d is the third proportional to a and b.
- If $a:b::b:c$, then a, b and c are in continued proportion.

- Theorem of Invertendo: If $a:b::c:d$ then $b:a::d:c$
- Theorem of Alternendo: If $a:b::c:d$ then $a:c::b:d$
- Theorem of Componendo: If $a:b::c:d$ then $a+b:b::c+d:d$
- Theorem of Dividendo: If $a:b::c:d$ then $a-b:b::c-d:d$
- Theorem of Componendo-Dividendo:
  If $a:b::c:d$ then $a+b:a-b::c+d:c-d$
- If a variable varies directly as the product of several others, we say that it varies jointly as the others.

REVIEW EXERCISE 3

1. Encircle the correct option.

(i) In a proportion $a:b::c:d$, a and d are called

(a) extreme (b) mean
(c) third proportional (d) mean proportional

(ii) In a ratio $a:b$, b is called

(a) consequent (b) antecedent (c) relation (d) ratio

(iii) If $x+13:x+7$ and $4:5$ are equal, then x is:

(a) 17 (b) −37 (c) 37 (d) −17

(iv) If $4x-7y=2x-3y$, then $x:y$ is equal to:

(a) 1 : 2 (b) 2 : 1 (c) 2 : 3 (d) 3 : 2

(v) Mean proportional of $16x^4y^4$ and z^8 is equal to

(a) $4x^2y^2z^4$ (b) $4y^2z^4$ (c) $16x^4y^4z^8$ (d) $\pm 4x^2y^2z^4$

(vi) Find x, if 12, x, and 3 are continued proportion

(a) 6 (b) 36 (c) ±6 (d) −6

(vii) If $a^2 \propto \frac{1}{b^5}$ then

(a) $a^2 = \frac{k}{b^5}$ (b) $a^2 = kb^5$ (c) $a^2 = -kb^5$ (d) $b^5 = ka^2$

(viii) If $a \propto b$ and $a \propto \frac{1}{c}$ then

(a) $b = c$ (b) $bc = 1$ (c) $bc = -1$ (d) $b = -c$

(ix) If $a:b=c:d$, then dividendo property is

(a) $a-b:b=c-d:d$ (b) $a+b:b=c+d:d$
(c) $a+b:a-b=c+d:c-d$ (d) $a-b:c=c-d:b$

(x) If x varies inversely as y^2, then k is:

(a) xy (b) x^2y (c) xy^2 (d) x^2y^2

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